LANMARK UNIVERSITY OMU-ARAN, KWARA STATE

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE: MECHANICS OF MACHINE (MCE 322).

LECTURER: ENGR. IBIKUNLE 'ROTIMI ADEDAYO

SIMPLE HARMONIC MOTION.

Introduction

Consider a particle moving round the circumference of a circle in an anticlockwise direction, with a constant angular velocity, as shown in Fig.1.

Let '**P**' be the position of the particle at any instant, and '**N**' is the projection of **P** on the diameter **XX'** of the circle.

When the point **P** moves round the circumference of the circle from **X** to **Y**, then **N** moves from **X** to **O**.

When **P** moves from **Y** to **X'**, then **N** moves from **O** to **X'**. Similarly when **P** moves from **X'** to **Y'**, then **N** moves from **X'** to **O** and finally when **P** moves from **Y'** to **X**, then **N** moves from **O** to **X**.

Hence, as **P** completes one revolution, the point **N** completes one vibration about the point **O**. This to and fro motion of **N** is known as ***simple harmonic motion*** (S.H.M.).

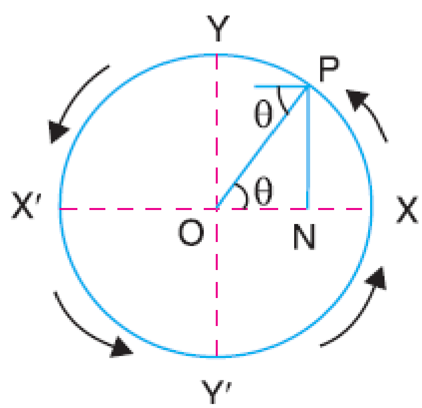


Fig.1.

1.2 Velocity and Acceleration of a Particle Moving with S.H.M

Let us consider a particle, moving round the circumference of a circle of radius **r**, with a uniform angular velocity **ω** rad/s, as shown in Fig.2.

Let **P** be any position of the particle after **t** seconds and **Θ** be the angle turned by the particle in *t* seconds. We know that **Θ = ω.t**

If **N** is the projection of **P** on the diameter **XX'**, then displacement of **N** from its mean position **O** is, always directed towards the Centre **O**; so that the motion of **N** is simple harmonic.

$$\mathbf{x = r.\cos\Theta = r.\cos.\omega t} \quad \ldots (i)$$

The velocity of **N** is the component of the velocity of **P** parallel to **XX '**,

$$\textit{i.e.} \quad \mathbf{V_N = v \sin\Theta = \omega.r \sin\Theta = \omega\sqrt{(r^2 - x^2)}} \quad \ldots (ii)$$

$$\ldots[\, v = r.\,\omega],\ \text{and } r \sin\Theta = NP = \sqrt{(r^2 - x^2)}.$$

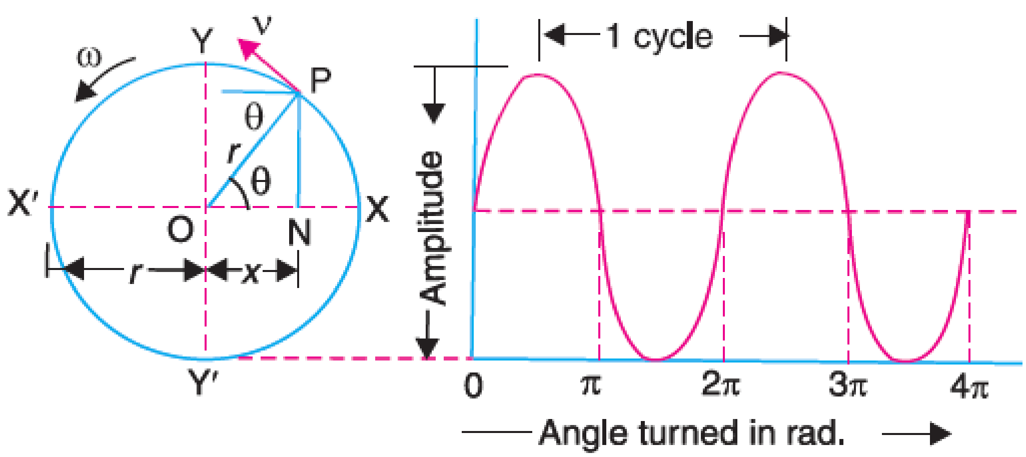


Fig.2. Velocity and acceleration of particle.

In general, a body is said to move or vibrate with simple harmonic motion, if it satisfies the following two conditions

1. ***Its acceleration is always directed towards the centre, known as point of reference or mean position;***

2. ***Its acceleration is proportional to the distance from that point.***

Movements of a ship up and down in a vertical plane about transverse axis (called Pitching) and about longitude (called rolling) are in Simple Harmonic Motion.

1.3 Differential Equation of Simple Harmonic Motion

We have discussed in the previous article that the displacement of N from its mean position O is

$$x = r.\cos\theta = r.\cos\omega t \qquad \dots (i)$$

Differentiating equation (*i*), we have velocity of N,

$$\frac{dx}{dt} = v_N = r.\omega\sin\omega t \qquad \dots (ii)$$

Again differentiating equation (*ii*), we have acceleration of N,

$$^{*}\frac{d^2x}{dt^2} = a_N = -r.\omega.\ \omega\cos\ \omega t = -\omega^2.r\cos\ \omega t = -\omega^2.x \qquad \dots (iii)$$

$$\dots (\because\ r\cos\omega t = x)$$

or $$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

This is the standard differential equation for simple harmonic motion of a particle. The solution of this differential equation is

$$x = A\cos\omega\, t + B\sin\omega\, t \qquad \dots (iv)$$

where A and B are constants to be determined by the initial conditions of the motion.

In Fig.2, when $t = 0$, $x = r$ *i.e.* when points P and N lie at X, we have from equation **(*iv*)**, $A = r$

Differentiating equation **(*iv*)**, $\frac{dx}{dt} = -A\ .\ \omega.\sin\omega.t + B.\omega\cos\omega t$

When $t = 0$, $\frac{dx}{dt} = 0$, therefore, from the above equation, $B = 0$. Now the equation (*iv*) becomes

$$x = r\cos\omega\, t \qquad \dots \text{[Same as equation } (i)]$$

The equations (*ii*) and (*iii*) may be written as

$$\frac{dx}{dt} = v_N = -\omega.r\ \sin\ \omega t = \omega.r\ \cos\ (\omega t + \pi/2)$$

and $$\frac{d^2x}{dt^2} = a_N = -\omega^2.r\ \cos\ \omega t = \omega^2.r\cos(\omega t + \pi)$$

These equations show that the velocity leads the displacement by 90° and acceleration leads the displacement by 180°.

* The negative sign shows that the direction of acceleration is opposite to the direction in which x increases, *i.e.* the acceleration is always directed towards the point O.

1.4. Terms Used in Simple Harmonic Motion

The following terms, commonly used in simple harmonic motion, are important from the subject point of view.

1. *Amplitude.* It is the maximum displacement of a body from its mean position. In Fig. 2, **OX** or **OX'** is the amplitude of the particle **P**. The amplitude is always equal to the radius of the circle.

2. *Periodic time*. It is the time taken for one complete revolution of the particle.

∴ Periodic time, $t_p = 2\pi/\omega$ seconds

$$a = \omega^2 . x \quad \text{or} \quad \omega^2 = \frac{a}{x} \quad \text{or} \quad \omega = \sqrt{\frac{a}{x}}$$

$$\therefore \qquad t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{x}{a}} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} \text{ seconds}$$

It is thus obvious, that the periodic time is independent of amplitude.

3. *Frequency*. It is the number of cycles per second and is the reciprocal of time period, t_p.

$$\therefore \qquad \text{Frequency, } n = \frac{\omega}{2\pi} = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{a}{x}} \text{ Hz}$$

Type equation here.**Notes: 1.** In S.I. units, the unit of frequency is hertz (briefly written as Hz) which is equal to one cycle per second.

2. When the particle moves with angular simple harmonic motion, then the periodic time

$$t_p = 2\pi \sqrt{\left\{\frac{\text{Angular displacement}}{\text{Angular acceleration}}\right\}} = 2\pi\sqrt{\frac{\theta}{\alpha}} \text{ sec.}$$

and $\qquad$ *frequency*, $n = \frac{1}{2\pi}\sqrt{\frac{\propto}{\theta}}$ HZ

Example 1.1. *The piston of a steam engine moves with simple harmonic motion. The crank rotates at 120 r.p.m. with a stroke of 2 metres. Find the velocity and acceleration of the piston, when it is at a distance of 0.75 metre from the centre.*

Solution. Given: N =120 r.p.m. or $\omega = 2\pi\times120/60 = 4\pi$ rad/s; $2r = 2$ m or $r = 1$ m; $x = 0.75$ m

Velocity of the piston

We know that velocity of the piston, $V = \omega\sqrt{r^2 - x^2} = 4\pi\sqrt{1 - (0.75)^2} = 8.31$ m/s.

Acceleration of the piston

We also know that acceleration of the piston,

$a = \omega^2 . x = (4\pi)^2\, 0.75 = 118.46$ m/s^2 **Ans.**

Example 1.2. *A point moves with simple harmonic motion. When this point is 0.75metre from the mid path, its velocity is 11 m/s and when 2 metres from the centre of its path its velocity is 3 m/s. Find its angular velocity, periodic time and its maximum acceleration.*

Solution. Given : When $x = 0.75$ m, $v = 11$ m/s ; when $x = 2$ m, $v = 3$ m/s

Angular velocity

Let ω = Angular velocity of the particle, and

r = Amplitude of the particle.

We know that velocity of the point when it is 0.75 m from the mid path (v),

$$11 = \omega\sqrt{r^2 - x^2} = \omega\sqrt{r^2 - (0.75)^2} \qquad \ldots (i)$$

Similarly, velocity of the point when it is 2 m from the centre (v),

$$3 = \omega\sqrt{r^2 - 2^2} \qquad \ldots (ii)$$

Dividing equation (*i*) by equation (*ii*),

$$\frac{11}{3} = \frac{\omega\sqrt{r^2 - (0.75)^2}}{\omega\sqrt{r^2 - 2^2}} = \frac{\sqrt{r^2 - (0.75)^2}}{\sqrt{r^2 - 2^2}}$$

Squaring both sides, $\frac{121}{9} = \frac{r^2 - 0.5625}{r^2 - 4}$

$121\, r^2 - 484 = 9r^2 - 5.06$ or $112\, r^2 = 478.94$

$\therefore$ $r2 = 478.94 / 112 = 4.276$ or $r = 2.07$ m

Substituting the value of r in equation **(*i*)**,

Solution. Given : When $x = 0.75$ m, $v = 11$ m/s ; when $x = 2$ m, $v = 3$ m/s

Angular velocity

Let ω = Angular velocity of the particle, and

r = Amplitude of the particle.

We know that velocity of the point when it is 0.75 m from the mid path (v),

$$11 = \omega\sqrt{r^2 - x^2} = \omega\sqrt{r^2 - (0.75)^2} \qquad \ldots (i)$$

Similarly, velocity of the point when it is 2 m from the centre (v),

$$3 = \omega\sqrt{r^2 - 2^2} \qquad \ldots (ii)$$

Dividing equation (*i*) by equation (*ii*),

$$\frac{11}{3} = \frac{\omega\sqrt{r^2 - (0.75)^2}}{\omega\sqrt{r^2 - 2^2}} = \frac{\sqrt{r^2 - (0.75)^2}}{\sqrt{r^2 - 2^2}}$$

Squaring both sides,

$$\frac{121}{9} = \frac{r^2 - 0.5625}{r^2 - 4}$$

$$121\, r^2 - 484 = 9r^2 - 5.06 \quad \text{or} \quad 112\, r^2 = 478.94$$

$$\therefore \quad r^2 = 478.94 / 112 = 4.276 \quad \text{or} \quad r = 2.07 \text{ m}$$

Substituting the value of r in equation (*i*),

$$11 = \omega\sqrt{(2.07)^2 - (0.75)^2} = 1.93\,\omega$$

$$\therefore \quad \omega = 11/1.93 = 5.7 \text{ rad/s} \ \textbf{Ans.}$$

Periodic time

We know that periodic time,

$$t_p = 2\pi / \omega = 2\pi / 5.7 = 1.1 \text{ s} \ \textbf{Ans.}$$

Maximum acceleration

We know that maximum acceleration,

$$a_{max} = \omega^2.r = (5.7)^2\, 2.07 = 67.25 \text{ m/s}^2 \ \textbf{Ans.}$$

1.5. Simple Pendulum

A simple pendulum, in its simplest form, consists of heavy bob suspended at the end of a light inextensible and flexible string. The other end of the string is fixed at ***O***, as shown in Fig. 3. Let ***L*** = Length of the string,

m = Mass of the bob in kg,

W = Weight of the bob in newtons = ***m.g,*** and

Θ = Angle through which the string is displaced.

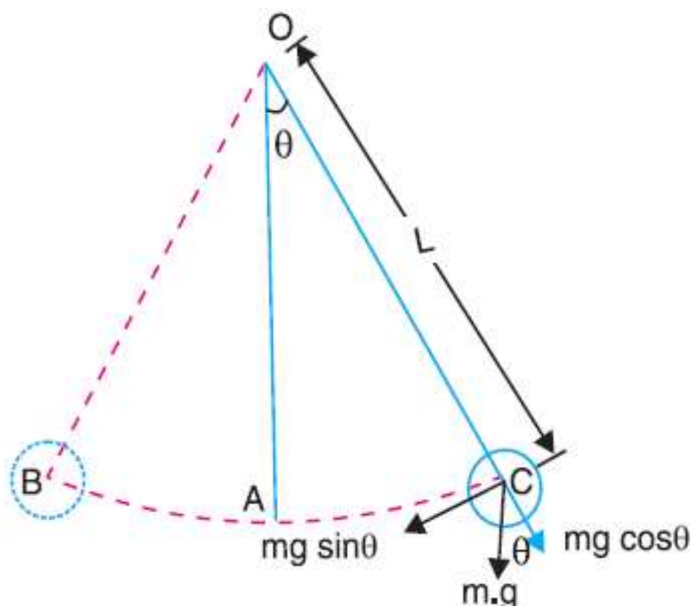


Fig .3. Simple pendulum.

When the bob is at *A*, the pendulum is in equilibrium position. If the bob is brought to *B* or *C* and released, it will start oscillating between the two positions *B* and *C*, with *A* as the mean position.

It has been observed that if the angle Θ is very small (less than 4°), the bob will have simple harmonic motion. Now, the couple tending to restore the bob to the equilibrium position or restoring torque,

$$T = m.g \sin \Theta \times L$$

Since angle Θ is very small, therefore sin Θ = Θ radians.

$$\therefore \quad T = m.g.L.\Theta$$

We know that the mass moment of inertia of the bob about an axis through the point of suspension,

$$I = \text{mass} \times (\text{length})^2 = m.L^2$$

∴ Angular acceleration of the string,

$$\propto = \frac{T}{I} = \frac{m.g.L.\theta}{m.L^2} = \frac{g.\theta}{L} \text{ or } \frac{\theta}{\alpha} = \frac{L}{g}$$

i.e. $$\frac{\text{Angular displacement}}{\text{Angular acceleration}} = \frac{L}{g}$$

We know that the periodic time,

$$t_p = 2\pi \sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi \sqrt{\frac{L}{g}} \qquad \dots (i)$$

and frequency of oscillation,

$$n = \frac{1}{t_p} = \frac{1}{2\pi} \sqrt{\frac{g}{L}} \qquad \dots (ii)$$

From above we see that the periodic time and the frequency of oscillation of a simple pendulum depends only upon its length and acceleration due to gravity. The mass of the bob has no effect on it.

Notes : 1. The motion of the bob from one extremity to the other (*i.e.* from *B* to *C* or *C* to *B*) is known as **beat** or **swing**. Thus one beat = $\frac{1}{2}$ oscillation.

$\therefore$ Periodic time for one beat = $\pi \sqrt{L/g}$

2. A pendulum, which executes one beat per second (*i.e.* one complete oscillation in two seconds) is known as a **second's pendulum**.

1.6. Laws of Simple Pendulum

The following laws of a simple pendulum are important from the subject point of view:

1. *Law of isochronism*. It states, "The time period (t_p) of a simple pendulum does not depend upon its amplitude of vibration and remains the same, provided the angular amplitude (Θ) does not exceed 4°."

2. *Law of mass*. It states, "The time period (t_p) of a simple pendulum does not depend upon the mass of the body suspended at the free end of the string."

3. *Law of length*. It states, "The time period (t_p) of a simple pendulum is directly proportional to $\sqrt{L}$, where *L* is the length of the string."

4. *Law of gravity*. It states, "The time period (t_p) of a simple pendulum is inversely proportional to $\sqrt{g}$, where *g* is the acceleration due to gravity."

Note: The above laws of a simple pendulum are true from the equation of the periodic time *i.e.*

$$t_p = 2\pi\sqrt{L}/g$$

1.7. Closely-coiled Helical Spring

Consider a closely-coiled helical spring, whose upper end is fixed, as shown in Fig. 4. Let a body be attached to the lower end. Let *A A* be the equilibrium position of the spring, after the mass is attached. If the spring is stretched up to *BB* and then released, the mass will move up and down with simple harmonic motion.

Let ***m*** = Mass of the body in kg,

W = Weight of the body in newtons = ***m.g,***

x = Displacement of the load below equilibrium position in metres,

s = Stiffness of the spring in N/m *i.e.* restoring force per unit displacement from the equilibrium position,

a = Acceleration of the body in m/s^2.

We know that the deflection of the spring, $\delta = \frac{m.g}{s}$ ... (*i*)

Then disturbing force = ***m.a***

And restoring force = ***s.x*** ... (*ii*)

Equating equations (*i*) and (*ii*), ***m.a*** = ***s.x**** or $= \frac{x}{a} = \frac{m}{s}$

The differential equation for the motion of the spring is

$$m\frac{d^2x}{dt^2} = -\,s\,x \text{ or } \frac{d^2x}{dt^2} = \frac{-sx}{m} \quad \text{.... (Here } \omega^2 = \frac{s}{m}\text{)}$$

The – ve sign indicates that the restoring force *s.x* is opposite to the direction of disturbing force.

We know that periodic time,

$$t_p = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{x}{a}}$$

$$= 2\pi\sqrt{\frac{m}{s}} = 2\pi\sqrt{\frac{\delta}{g}} \quad \ldots \left(\because \delta = \frac{mg}{s}\right)$$

and frequency, $n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$

Note: If the mass of the spring (m_1) is also taken into consideration, then the periodic time,

$$t_p = 2\pi\sqrt{\frac{m + m_1/3}{s}} \text{ seconds,}$$

and frequency, $n = \frac{1}{2\pi}\sqrt{\frac{s}{m + m_1/3}}$ Hz

When we stretch a spring with a mass on the end and let it go, the mass will oscillate back and forth (if there is no friction). This oscillation is called ***S.H.M.***

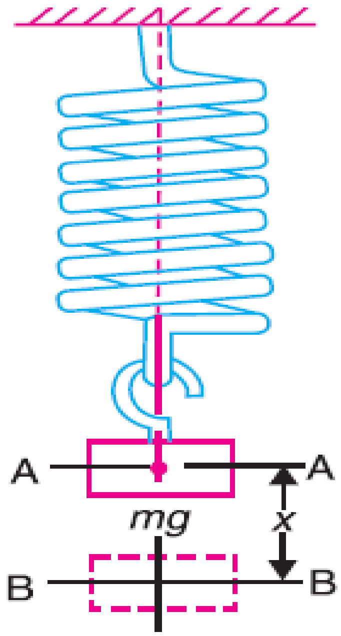


Fig. 4 Closely - coiled helical spring.

Example 1.3. *A helical spring, of negligible mass, and which is found to extend 0.25 mm under a mass of 1.5 kg, is made to support a mass of 60 kg. The spring and the mass system is displaced vertically through 12.5 mm and released. Determine the frequency of natural vibration of the system. Find also the velocity of the mass, when it is 5 mm below its rest position.*

Solution. Given: m = 60 kg ; r = 12.5 mm = 0.0125 m ; x = 5 mm = 0.005 mSince a mass of 1.5 kg extends the spring by 0.25 mm, therefore a mass of 60 kg will extend the spring by an amount,

$$\delta = \frac{0.25}{1.5} \times 60 = 10 \text{ mm} = 0.01 \text{ m}$$

Frequency of the system

We know that frequency of the system,

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{0.01}} = 4.98 \text{ Hz } \textbf{Ans}$$

Velocity of the mass

Let v = Linear velocity of the mass.

We know that angular velocity,

$$\omega^* = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.01}} = 31.32 \text{ rad/s}$$

and

$$v = \omega\sqrt{r^2 - x^2} = 31.32\sqrt{(0.0125)^2 - (0.005)^2} = 0.36 \text{ m/s } \textbf{Ans.}$$

1.8. Compound Pendulum

When a rigid body is suspended vertically, and it oscillates with a small amplitude under the action of the force of gravity, the body is known as ***compound pendulum***, as shown in Fig.5.

Let $\boldsymbol{m}$ = Mass of the pendulum in kg,

$\boldsymbol{W}$ = Weight of the pendulum in newtons = $\boldsymbol{m.g}$,

* We know that periodic time,

$$t_p = \frac{2\pi}{\omega} \text{ or } \omega = \frac{2\pi}{t_p} = 2\pi \times n = 2\pi \times 4.98 = 31.3 \text{ rad/s}$$

$(\because \; n = 1/tp)$

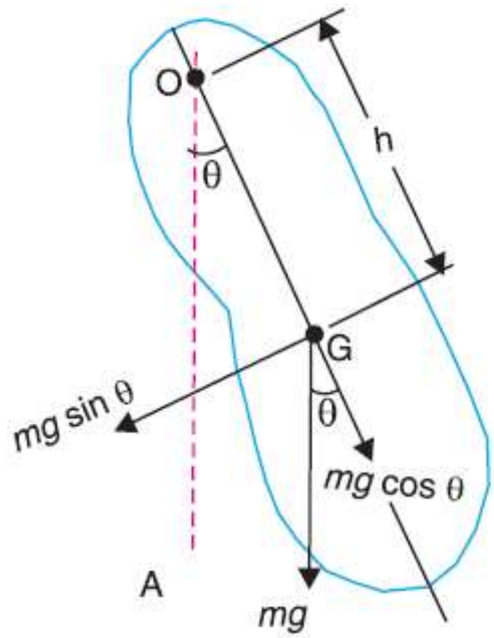


Fig. 5

$\boldsymbol{k_G}$ = Radius of gyration about an axis through the centre of gravity G and perpendicular to the plane of motion, and

$\boldsymbol{h}$ = Distance of point of suspension $\boldsymbol{O}$ from the centre of gravity $\boldsymbol{G}$ of the body.

If the pendulum is given a small angular displacement $\boldsymbol{\theta}$, then the couple tending to restore the pendulum to the equilibrium position $\boldsymbol{OA}$,

$$T = mg \sin \theta \times h = m\,g\,h \sin \theta$$

Since θ is very small, therefore substituting $\sin \theta = \theta$ radians, we get

$$T = m\,g\,h\,\theta$$

Now, the mass moment of inertia about the axis of suspension O,

$$I = I_G + m.h^2 = m\left(k_G^2 + h^2\right) \quad \text{. . . (By parallel axis theorem)}$$

∴ Angular acceleration of the pendulum,

$$\alpha = \frac{T}{I} = \frac{mgh\,\theta}{m(k_G^2 + h^2)} = \frac{gh\,\theta}{k_G^2 + h^2} = \text{constant} \times \theta$$

We see that the angular acceleration is directly proportional to angular displacement, therefore the pendulum executes simple harmonic motion.

$$\therefore \qquad \frac{\theta}{\alpha} = \frac{k_G^2 + h^2}{g.h}$$

We know that the periodic time,

$$t_p = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{\theta}{\alpha}}$$

$$= 2\pi\sqrt{\frac{k_G^2 + h^2}{g.h}} \quad \text{... } (i)$$

and frequency of oscillation, $n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{g.h}{k_G^2 + h^2}}$ ... ***(ii)***

Notes: 1. Comparing this equation with equation ***(ii)*** of simple pendulum, we see that the equivalent length of a simple pendulum, which gives the same frequency as compound pendulum, is L= $\frac{K^2 + h^2}{h} = \frac{K^2{}_G}{h} + h$

2. Since the equivalent length of simple pendulum (L) depends upon the distance between the point of suspension and the centre of gravity (G), therefore L can be changed by changing the position of point of suspension.

This will, obviously, change the periodic time of a compound pendulum. The periodic time will be minimum if L is minimum. For L to be minimum, the differentiation of L with respect to h must be equal to zero, *i.e.*

$$\frac{dL}{dh} = 0 \text{ or } \frac{d}{dh} = \left(\frac{K^2{}_G}{h} + h\right) = 0$$

$$\frac{K^2{}_G}{h} + 1 = 0 \text{ or } K_G = \text{h}$$

Thus the periodic time of a compound pendulum is minimum when the distance between the point of suspension and the centre of gravity is equal to the radius of gyration of the body about its centre of gravity.

∴ Minimum periodic time of a compound pendulum,

$$t_{P_{(\min)}} = 2\pi\sqrt{\frac{2K_G}{g}} \qquad \text{. . . [Substituting } h = k_G \text{ in equation } (i)\text{]}$$

Example 4.4. *A uniform thin rod, as shown in Fig. 4.7, has a mass of 1 kg and carries a concentrated mass of 2.5 kg at B. The rod is hinged at A and is maintained in the horizontal position by a spring of stiffness 1.8 kN/m at C. Find the frequency of oscillation, neglecting the effect of the mass of the spring.*

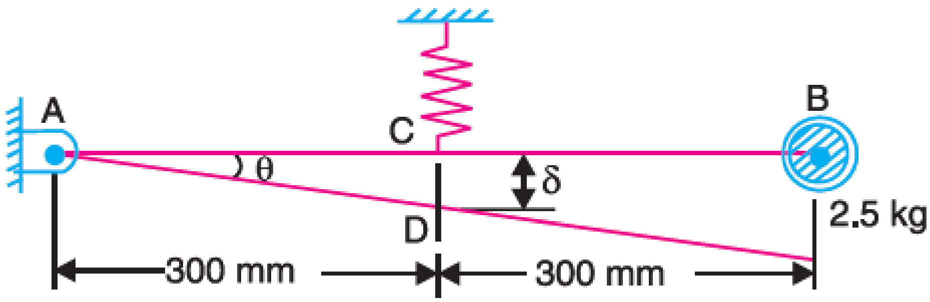


Fig. 6.

Solution. Given : $m = 1$ kg ; $m_1 = 2.5$ kg ; $s = 1.8$ kN/m $= 1.8 \times 10^3$ N/m

We know that total length of rod,

$$l = 300 + 300 = 600 \text{ mm} = 0.6 \text{ m}$$

∴ Mass moment of inertia of the system about A,

I_A = Mass moment of inertia of 1 kg about A + Mass moment of interia of 2.5 kg about A

$$= \frac{m.l^2}{3} + m_1 l^2 = \frac{1(0.6)^2}{3} + 2.5\ (0.6)^2 = 1.02 \text{ kg-m}^2$$

If the rod is given a small angular displacement θ and then released, the extension of the spring,

$$\delta = 0.3 \sin\theta = 0.3\theta \text{ m}$$

. . . (∵ θ is very small, therefore substituting sin θ = θ)

∴ Restoring force $= s.\delta = 1.8 \times 10^3 \times 0.3\ \theta = 540\ \theta$ N

and restoring torque about A $= 540\ \theta \times 0.3 = 162\ \theta$ N-m ... (*i*)

We know that disturbing torque about A

$= I_A \times \alpha = 1.02\alpha$ N-m ... (*ii*)

Equating equations (*i*) and (*ii*),

$$1.02\ \alpha = 162\ \theta \quad \text{or} \quad \alpha / \theta = 162 / 1.02 = 159$$

We know that frequency of oscillation,

$$n = \frac{1}{2\pi}\sqrt{\frac{\alpha}{\theta}} = \frac{1}{2\pi}\sqrt{159}$$

$$= 2.01 \text{ HZ.}$$

Example 4.5. *A small flywheel of mass 85 kg is suspended in a vertical plane as a compound pendulum. The distance of centre of gravity from the knife edge support is 100 mm and the flywheel makes 100 oscillations in 145 seconds. Find the moment of inertia of the flywheel through the centre of gravity.*

Solution. Given: $m = 85$ kg; $h = 100$ mm $= 0.1$ m

Since the flywheel makes 100 oscillations in 145 seconds, therefore frequency of oscillation, $n = 100/145 = 0.69$ Hz

Let L = Equivalent length of simple pendulum, and

k_G = Radius of gyration through C.G.

We know that frequency of oscillation (n),

$$0.69 = \frac{1}{2\pi}\sqrt{\frac{g}{L}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{L}} = \frac{0.5}{\sqrt{L}}$$

$$\therefore \quad \sqrt{L} = 0.5/0.69 = 0.7246 \quad \text{or} \quad L = 0.525 \text{ m}$$

We also know that equivalent length of simple pendulum (L),

$$0.525 = \frac{k_G^2}{h} + h = \frac{k_G^2}{0.1} + 0.1 = \frac{k_G^2 + (0.1)^2}{0.1}$$

$$k_G^2 = 0.525 \times 0.1 - (0.1)^2 = 0.0425 \text{ m}^2$$

and moment of inertia of the flywheel through the centre of gravity,

$I = m.k^2_G = 85 \times 0.0425 = 3.6$ kg-m^2 **Ans**.

Example 4.9. *A small connecting rod of mass 1.5 kg is suspended in a horizontal plane by two wires 1.25 m long. The wires are attached to the rod at points 120 mm on either side of the centre of gravity. If the rod makes 20 oscillations in 40 seconds, find the radius of gyration and the mass moment of inertia of the rod about a vertical axis through the centre of gravity.*

Solution. Given: $m = 1.5$ kg; $l = 1.25$ m; $x = y = 120$ mm $= 0.12$ m

Since the rod makes 20 oscillations in 40 s, therefore frequency of oscillation,

$n = 20/40 = 0.5$ Hz

Radius of gyration of the connecting rod

Let k_G = Radius of gyration of the connecting rod.

We know that frequency of oscillation (n),

$$0.5 = \frac{1}{2\pi\, K_G}\sqrt{\frac{g.x.y}{I}} = \frac{1}{2\pi\, K_G}\sqrt{\frac{9.81\times 0.12\times 0.12}{1.25}} = \frac{0.0535}{K}$$

$\therefore \quad K_G = 0.0535/0.5 = 0.107\text{m} = 107\text{mm}.$

Mass moment of inertia of the connecting rod

We know that mass moment of inertia,

$$I = m\,(k_G)^2 = 1.5\,(0.107)^2 = 0.017 \text{ kg-m2}$$ **Ans.**

TUTORIALS .1.

1. A particle, moving with simple harmonic motion, performs 10 complete oscillations per minute and its speed, when at a distance of 80 mm from the centre of oscillation is 3/5 of the maximum speed.
Find the amplitude, the maximum acceleration and the speed of the particle, when it is 60 mm from the centre of the oscillation. **[Ans. 100 mm; 109.6 mm/s2; 83.76 mm/s]**

2. A piston, moving with a simple harmonic motion, has a velocity of 8 m/s, when it is 1 metre from the centre position and a velocity of 4 m/s, when it is 2 metres from the centre. Find: 1. Amplitude, 2.Periodic time, 3. Maximum velocity, and 4. Maximum acceleration. **[Ans. 2.236 m; 1.571 s; 8.94 m/s; 35.77 m/s2]**

3. The plunger of a reciprocating pump is driven by a crank of radius 250 mm rotating at 12.5 rad/s.
Assuming simple harmonic motion, determine the maximum velocity and maximum acceleration of the plunger. **[Ans. 3.125 m/s; 39.1 m/s2]**

4. A part of a machine of mass 4.54 kg has a reciprocating motion which is simple harmonic in character.
It makes 200 complete oscillations in 1 minute. Find: 1. the accelerating force upon it and its velocity when it is 75 mm, from mid-stroke; 2. the maximum accelerating force, and 3. the maximum velocity if its total stroke is 225 mm *i.e.* if the amplitude of vibration is 112.5 mm. **[Ans. 149.5 N ; 1.76 m/s ; 224 N ; 2.36 m/s]**

5. A helical spring of negligible mass is required to support a mass of 50 kg. The stiffness of the spring is 60 kN/m. The spring and the mass system is displaced vertically by 20 mm below the equilibrium position and then released. Find: 1. the frequency of natural vibration of the system; 2. the velocity and acceleration of the mass when it is 10 mm below the rest position. **[Ans. 5.5 Hz; 0.6 m/s ; 11.95 m/s2]**

6. A spring of stiffness 2 kN/m is suspended vertically and two equal masses of 4 kg each are attached to the lower end. One of these masses is suddenly removed and the system oscillates.

Determine: 1. The amplitude of vibration, 2. the frequency of vibration, 3. the velocity and acceleration of the mass when passing through half amplitude position, and 4. kinetic energy of the vibration in joules.
[Ans. 0.019 62 m; 3.56 Hz; 0.38 m/s, 4.9 m/s^2; 0.385 J]

7. A vertical helical spring having a stiffness of 1540 N/m is clamped at its upper end and carries a mass of 20 kg attached to the lower end. The mass is displaced vertically through a distance of 120 mm and released. Find: 1. Frequency of oscillation; 2. Maximum velocity reached; 3. Maximum acceleration; and 4. Maximum value of the inertia force on the mass. **[Ans. 1.396 Hz; 1.053 m/s; 9.24 m/s^2; 184.8 N]**

8. A small flywheel having mass 90 kg is suspended in a vertical plane as a compound pendulum. The distance of centre of gravity from the knife edge support is 250 mm and the flywheel makes 50 oscillations in 64 seconds. Find the moment of inertia of the flywheel about an axis through the centre of gravity. **[Ans. 3.6 kg-m2].**

2.0 VIBRATION.

2.1. Introduction

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute a ***vibratory motion*.** This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position. When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilibrium position. In this way, the vibratory motion is repeated indefinitely.

2.2. Terms Used in Vibratory Motion

The following terms are commonly used in connection with the vibratory motions:

1. *Period of vibration or time period*. It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

2. *Cycle*. It is the motion completed during one time period.

3. *Frequency*. It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

2.3. Types of Vibratory Motion

The following types of vibratory motion are important from the subject point of view:

1. *Free or natural vibrations*. When no external force acts on the body, after giving it an initial displacement, then the body is said to be under ***free or natural vibrations*.** The frequency of the free vibrations is called ***free or natural frequency*.**

2. *Forced vibrations*. When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations*.** The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

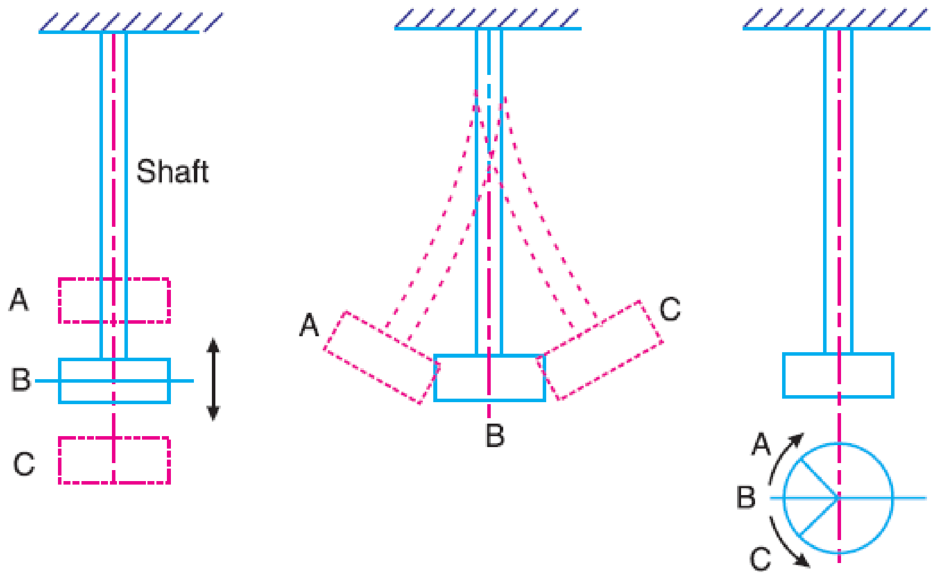
Note: When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

3. *Damped vibrations*. When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration*.** This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

2.4. Types of Free Vibrations

The following three types of free vibrations are important from the subject point of view:

1. **Longitudinal vibrations,** 2. **Transverse vibrations,** and 3. **Torsional vibrations.** Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig.7. This system may execute one of the three above mentioned types of vibrations.



B = Mean position ; *A* and *C* = Extreme positions.

2. (*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations

1. *Longitudinal vibrations*. When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig.7. (*a*), then the vibrations are known as ***longitudinal vibrations*.**

In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

2. *Transverse vibrations*. When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig.7. (*b*), then the vibrations are known as ***transverse vibrations*.** In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

3. *Torsional vibrations**. When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig.7 (*c*), then the vibrations are known as ***torsional vibrations*.**

In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

Note: If the limit of proportionality (*i.e.* stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, simple harmonic.

2.5. Natural Frequency of Free Longitudinal Vibrations

The natural frequency of the free longitudinal vibrations may be determined by the following three methods:

1. *Equilibrium Method*

Consider a constraint (*i.e.* spring) of negligible mass in an unstrained position, as shown in Fig. 8. (*a*).

Let ***s*** = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.

m = Mass of the body suspended from the constraint in kg,

W = Weight of the body in newtons = ***m.g***,

δ = Static deflection of the spring in metres due to weight ***W*** newtons, and

x = Displacement given to the body by the external force, in metres.

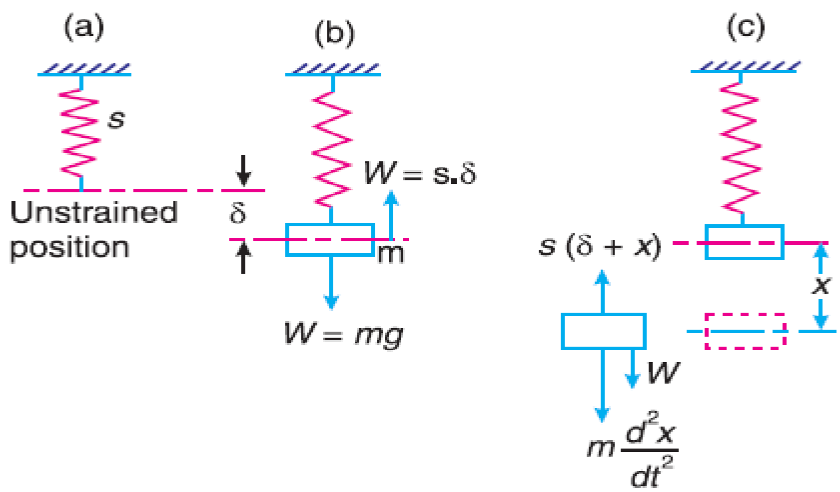


Fig.8. Natural frequency of free longitudinal vibrations.

In the equilibrium position, as shown in Fig.8. (*b*), the gravitational pull ***W* = *m.g*,** is balanced by a force of spring, such that ***W* = S.δ.**

Since the mass is now displaced from its equilibrium position by a distance ***x***, as shown in Fig. 8 (*c*), and is then released, therefore after time ***t***, Restoring force $= W - s(\delta + x) = W - s.\delta - s.x$

$= s.\delta - s.\delta - s.x = -s.x$ . . . ($\because W = s.\delta$) . . . (*i*)

. . . (Taking upward force as negative)

and Accelerating force = Mass × Acceleration

$= m \times \frac{d^2x}{dt^2}$ . . . (Taking downward force as positive) . . . (*ii*)

Equating equations (*i*) and (*ii*), the equation of motion of the body of mass m after time t is

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \ldots (iii)$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2 . x = 0 \qquad \ldots (iv)$$

Comparing equations (*iii*) and (*iv*), we have

$$\omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ . . . ($\because m.g = s.\delta$)

Taking the value of g as 9.81 m/s^2 and δ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz}$$

Note : The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{or} \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad \text{or} \quad \delta = \frac{W.l}{E.A}$$

where δ = Static deflection *i.e.* extension or compression of the constraint,

W = Load attached to the free end of constraint,

l = Length of the constraint,

E = Young's modulus for the constraint, and

A = Cross-sectional area of the constraint.

2. *Energy method*

We know that the kinetic energy is due to the motion of the body and the potential energy is with respect to a certain datum position which is equal to the amount of work required to move the body from the datum position.

In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero.

In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy and potential energy must be a constant quantity which is same at all the times. In other words,

$$\therefore \qquad \frac{d}{d_t}\,(K.E.+P.E) = 0$$

We know that kinetic energy,

$$K.E. = I/2.m.\left(\frac{d_x}{d_t}\right)^2$$

and potential energy,

$$P.E. = \left(\frac{0+s.x}{2}\right)x = \frac{1}{2}\times s.x^2$$

. . . ($\because$ *P.E.* = Mean force × Displacement)

$$\therefore \qquad \frac{d}{dt}\left[\frac{1}{2}\times m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\times s.x^2\right] = 0$$

$$\frac{1}{2}\times m\times 2\times\frac{dx}{dt}\times\frac{d^2x}{dt^2}+\frac{1}{2}\times s\times 2x\times\frac{dx}{dt} = 0$$

or $\quad m\times\frac{d^2x}{dt^2}+s.x = 0 \quad$ or $\quad \frac{d^2x}{dt^2}+\frac{s}{m}\times x = 0 \quad$ . . . (Same as before)

The time period and the natural frequency may be obtained as discussed in the previous method.

3. *Rayleigh's method*

In this method, the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position. Assuming the motion executed by the vibration to be simple harmonic, then

$$x = X \sin\omega.t \qquad \ldots (i)$$

where x = Displacement of the body from the mean position after time t seconds, and

X = Maximum displacement from mean position to extreme position.

Now, differentiating equation ***(i)***, we have

$$\frac{dx}{dt} = \omega \times X \cos\omega.t$$

Since at the mean position, $t = 0$, therefore maximum velocity at the mean position,

$$v = \frac{dx}{dt} = \omega.X$$

$\therefore$ Maximum kinetic energy at mean position

$$= \frac{1}{2} \times m.v^2 = \frac{1}{2} \times m.\omega^2.X^2 \qquad \ldots (ii)$$

and maximum potential energy at the extreme position

$$= \left(\frac{0 + s.X}{2}\right) X = \frac{1}{2} \times s.X^2 \qquad \ldots (iii)$$

Equating equations ***(ii)*** and ***(iii)***,

$$\frac{1}{2} \times m.\omega^2.X^2 = \frac{1}{2} \times s.X^2 \quad \text{or} \quad \omega^2 = \frac{s}{m}, \text{ and } \omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $t_p = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{s}{m}}$ . . . (Same as before)

and natural frequency, $f_n = \dfrac{1}{t_p} = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{s}{m}}$

Note: In all the above expressions, ω is known as **natural circular frequency** and is generally denoted by ω_n.

2.6. Natural Frequency of Free Transverse Vibrations

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight ***W***, as shown in Fig.9.

Let ***s*** = Stiffness of shaft,

δ = Static deflection due to weight of the body,

x = Displacement of body from mean position after time *t*.

m = Mass of body = ***W/g***

Restoring force = – ***s.x*** . . . **(*i*)**

and accelerating force = **m** $\times \frac{d^2 x}{d t^2}$ . . . **(*ii*)**

Equating equations **(*i*)** and **(*ii*)**, the equation of motion becomes

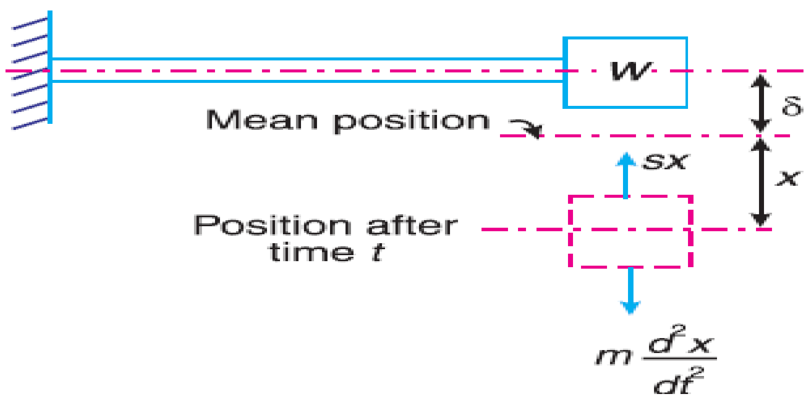


Fig. 9. Natural frequency of free transverse vibrations.

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (Same as before)}$$

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

$$\text{Time period,} \quad t_p = 2\pi\sqrt{\frac{m}{s}}$$

and natural frequency, $$f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

Note : The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on Strength of Materials, that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI} \text{ (in metres)}$$

where W = Load at the free end, in newtons,

l = Length of the shaft or beam in metres,

E = Young's modulus for the material of the shaft or beam in N/m^2, and

I = Moment of inertia of the shaft or beam in m^4.

Example 2.1. *A cantilever shaft 50 mm diameter and 300 mm long has a disc of mass 100 kg at its free end. The Young's modulus for the shaft material is 200 GN/m^2. Determine the frequency of longitudinal and transverse vibrations of the shaft.*

Solution. Given: d = 50 mm = 0.05 m; l = 300 mm = 0.3 m; m = 100 kg;
$E = 200\ GN/m^2 = 200 \times 10^9\ N/m^2$
We know that cross-sectional area of the shaft,
Solution. Given: d = 50 mm = 0.05 m; l = 300 mm = 0.03 m ; m = 100 kg ;
E = 200 GN/m2 = 200 ×109 N/m2
We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3}\,m^2$$

and moment of inertia of the shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.3 \times 10^{-6}\,m^4$$

Frequency of longitudinal vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l}{A.E} = \frac{100 \times 9.81 \times 0.3}{1.96 \times 10^{-3} \times 200 \times 10^9} = 0.751 \times 10^{-6}\ m$$

$$\ldots (\because\ W = m.g)$$

$\therefore$ Frequency of longitudinal vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.751 \times 10^{-6}}} = 575 \text{ Hz} \quad \textbf{Ans.}$$

Frequency of transverse vibration

We know that static deflection of the shaft,

$$\delta = \frac{W.l^3}{3E.I} = \frac{100 \times 9.81 \times (0.3)^3}{3 \times 200 \times 10^9 \times 0.3 \times 10^{-6}} = 0.147 \times 10^{-3} \text{ m}$$

$\therefore$ Frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta}} = \frac{0.4985}{\sqrt{0.147 \times 10^{-3}}} = 41 \text{ Hz} \quad \textbf{Ans.}$$

2.7. Effect of Inertia of the Constraint in Longitudinal and Transverse Vibrations

In deriving the expressions for natural frequency of longitudinal and transverse vibrations, we have neglected the inertia of the constraint *i.e*. shaft. We shall now discuss the effect of the inertia of the constraint, as below:

1. *Longitudinal vibration*

Consider the constraint whose one end is fixed and other end is free as shown in Fig.10.

Let m_1 = Mass of the constraint per unit length,

l = Length of the constraint,

m_C = Total mass of the constraint = $m_1.l$, and

v = Longitudinal velocity of the free end.

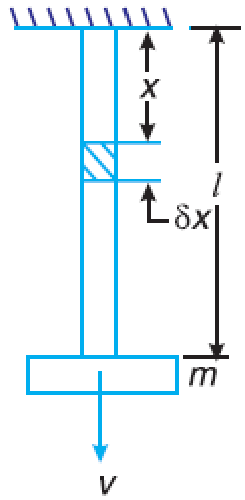


Fig. 10. Effect of inertia of the constraint in longitudinal vibrations.

Consider a small element of the constraint at a distance *x* from the fixed end and of Length δx.

∴ Velocity of the small element

$$= \frac{x}{l} \times v$$

and kinetic energy possessed by the element

$$= \frac{1}{2} \times \text{Mass (velocity)}^2$$

$$= \frac{1}{2} \times m_1 . \delta x \left(\frac{x}{l} \times v \right)^2 = \frac{m_1 . v^2 \, x^2}{2l^2} \times \delta x$$

∴ Total kinetic energy possessed by the constraint,

$$= \int_0^l \frac{m_1 . v^2 x^2}{2l^2} \times dx = \frac{m_1 . v^2}{2l^2} \left[\frac{x^3}{3} \right]_0^l$$

$$= \frac{m_1.v^2}{2l^2} \times \frac{l^3}{3} = \frac{1}{2} \times m_1.v^2 \times \frac{l}{3} = \frac{1}{2}\left(\frac{m_1.l}{3}\right)v^2 = \frac{1}{2}\left(\frac{m_C}{3}\right)v^2 \quad \ldots (i)$$

. . . (Substituting $m_1 . l = m_C$)

If a mass of $\frac{m_C}{3}$ is placed at the free end and the constraint is assumed to be of negligible mass, then

Total kinetic energy possessed by the constraint

$$= \frac{1}{2}\left(\frac{m_C}{3}\right)v^2 \qquad \ldots \text{[Same as equation } (i)] \ldots (ii)$$

Hence the two systems are dynamically same. Therefore, inertia of the constraint may be allowed for by adding one-third of its mass to the disc at the free end.

From the above discussion, we find that when the mass of the constraint m_C and the mass of the disc m at the end is given, then natural frequency of vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \frac{m_C}{3}}}$$

2. *Transverse vibration*

Consider a constraint whose one end is fixed and the other end is free as shown in Fig.11.

Let m_1 = Mass of constraint per unit length,

l = Length of the constraint,

m_C = Total mass of the constraint = $m_1.l$, and

v = Transverse velocity of the free end.

Consider a small element of the constraint at a distance x from the fixed end and of length δx. The velocity of this element is

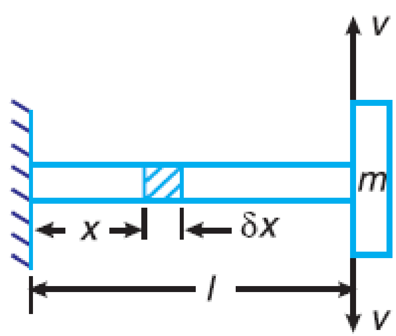


Fig.11. Effect of inertia of the constraint in transverse vibrations.

given by $\left[\dfrac{3l.x^2 - x^3}{2l^3} \times v\right]$.

$\therefore$ Kinetic energy of the element

$$= \frac{1}{2} \times m_1.\delta x\left(\frac{3l.x^2 - x^3}{2l^3} \times v\right)^2$$

and total kinetic energy of the constraint,

$$= \int_0^l \frac{1}{2} \times m_1 \left(\frac{3l.x^2 - x^3}{2l^3} \times v\right)^2 dx = \frac{m_1.v^2}{8l^6} \int_0^l (9l^2.x^4 - 6l.x^5 + x^6)\, dx$$

$$= \frac{m_1.v^2}{8l^6}\left[\frac{9l^2.x^5}{5} - \frac{6l.x^6}{6} + \frac{x^7}{7}\right]_0^l$$

$$= \frac{m_1.v^2}{8l^6}\left[\frac{9l^7}{5} - \frac{6l^7}{6} + \frac{l^7}{7}\right] = \frac{m_1.v^2}{8l^6}\left(\frac{33l^7}{35}\right)$$

$$= \frac{33}{280} \times m_1.l.v^2 = \frac{1}{2}\left(\frac{33}{140} \times m_1.l\right)v^2 = \frac{1}{2}\left(\frac{33}{140} \times m_C\right)v^2 \quad \ldots (i)$$

. . . (Substituting $m_1.l = m_C$)

If a mass of $\frac{33\, m_C}{140}$ is placed at the free end and the constraint is assumed to be of negligible mass, then

Total kinetic energy possessed by the constraint

$$= \frac{1}{2}\left(\frac{33\, m_C}{140}\right)v^2 \qquad \text{. . . [Same as equation } (i)]$$

Hence the two systems are dynamically same. Therefore the inertia of the constraint may be allowed for by adding $\frac{33}{140}$ of its mass to the disc at the free end.

From the above discussion, we find that when the mass of the constraint m_C and the mass of the disc m at the free end is given, then natural frequency of vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \frac{33\, m_C}{140}}}$$

Notes: 1. If both the ends of the constraint are fixed, and the disc is situated in the middle of it, then proceeding in the similar way as discussed above, we may prove that the inertia of the constraint may be allowed for by adding $\frac{13}{35}$ of its mass to the disc.

2. If the constraint is like a simply supported beam, then $\frac{17}{35}$ of its mass may be added to the mass of the disc.

2.8. Natural Frequency of Free Transverse Vibrations Due to a Point Load Acting Over a Simply Supported Shaft

Consider a shaft ***AB*** of length ***l***, carrying a point load ***W*** at ***C*** which is at a distance of l_1 from ***A*** and l_2 from ***B***, as shown in Fig. 12. A little consideration will show that when the shaft is deflected and suddenly released, it will make transverse vibrations. The deflection of the shaft is proportional to the load ***W*** and if the beam is deflected beyond the static equilibrium position then the load will vibrate with simple harmonic motion (as by a helical spring). If ™ is the static deflection due to load ***W***, then the natural frequency of the free transverse vibration is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\partial}} = \frac{0.4985}{\partial}\ \text{HZ} \qquad (g = 9.81\text{m/s}^2)$$

Some of the values of the static deflection for the various types of beams and under various load conditions are given in the following table.

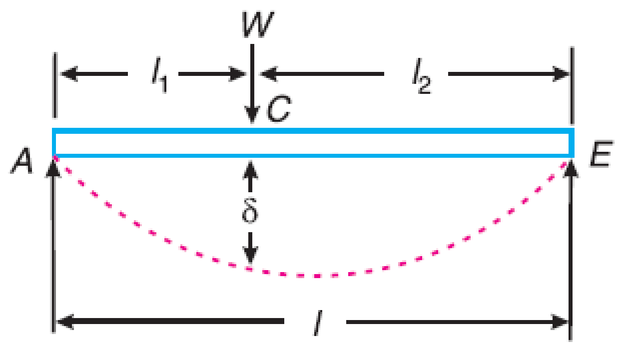


Fig.12. Simply supported beam with a point load.

Example 2.2. *A shaft of length 0.75 m, supported freely at the ends, is carrying a body of mass 90 kg at 0.25 m from one end. Find the natural frequency of transverse vibration. Assume*
E = 200 GN/m2 and shaft diameter = 50 mm
Solution. Given: l = 0.75 m; m = 90 kg; a = AC = 0.25 m; E = 200 GN/m^2 = 200 × 109 N/m2; d = 50 mm = 0.05 m
The shaft is shown in Fig.13.

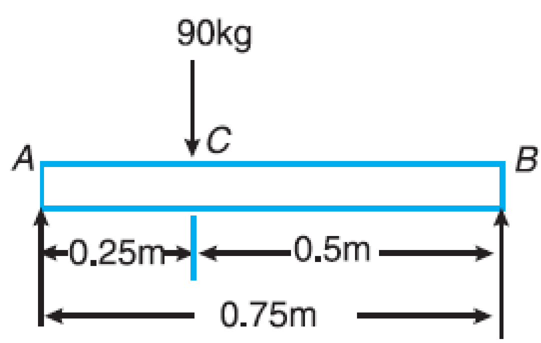


Fig. 13

We know that moment of inertia of the shaft,

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64}\ (0.05)^4 \mathrm{m}^4$$
$$= 0.307 \times 10^{-6}\ \mathrm{m}^4$$

and static deflection at the load point (*i.e.* at point *C*),

$$\delta = \frac{Wa^2b^2}{3EIl} = \frac{90 \times 9.81 \times 0.25^2 \times 0.5^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 0.75}$$

$$= 0.1 \times 10^{-3} \text{m} \qquad (\because b = BC = 0.5\ m)$$

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.495}{\sqrt{\partial}} = 49.85 \text{ Hz}$$

Example 2.3. *A flywheel is mounted on a vertical shaft as shown in Fig.14. The both ends of the shaft are fixed and its diameter is 50 mm. The flywheel has a mass of 500 kg. Find the natural frequencies of longitudinal and transverse vibrations. Take E = 200 GN/m^2.*

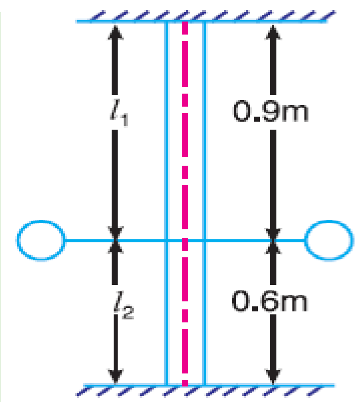


Fig. 14

Solution. Given: d = 50 mm = 0.05 m; m = 500 kg; E = 200 GN/m^2 = 200 × 10^9 N/m^2

We know that cross-sectional area of shaft,

$$A = \frac{\pi}{4} \times d^2 = \frac{\pi}{4}(0.05)^2 = 1.96 \times 10^{-3} \text{ m}^2$$

and moment of inertia of shaft,

$$I = \frac{\pi}{64} \times d^4 = \frac{\pi}{64}(0.05)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

Natural frequency of longitudinal vibration

Let m_1 = Mass of flywheel carried by the length l_1.

∴ $m - m_1$ = Mass of flywheel carried by length l_2.

We know that extension of length *l*1

$$\frac{W_1 l_1}{AE} = \frac{m_{1.g.} l_1}{AE} \qquad \text{...i}$$

Similarly, compression of length l_2 $= \dfrac{(w-w_{1)}\, l_2}{A.E}$

$$= - \frac{(m - m_1) \cdot g \cdot l_2}{A.E} \qquad \text{...ii}$$

Since extension of length $l1$ must be equal to compression of length $l2$, therefore equating equations ***(i)*** and ***(ii)***,

$$m_1 . l_1 = (m - m_1)\, l_2$$

$$m_1 \times 0.9 = (500 - m_1)\, 0.6$$

$$= 300 - 0.6 m_1 \text{ or } m_1 = 200 \text{ kg}$$

∴ Extension of length l_1, $$\partial = \frac{m_1 . g . \, l_1}{A.E} = \frac{200 \times 9.81 \times 0.9}{1.96 \times 10^{-3} \times 200 \times 10^{9}}$$

$$= 4.5 \times 10^{-6} \text{ m}$$

We know that natural frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\partial}} = \frac{0.4985}{\sqrt{4.5 \times 10^{-6}}}$$

$$= 235 \text{ Hz}$$

Natural frequency of transverse vibration

We know that the static deflection for a shaft fixed at both ends and carrying a point load is given by

$$\partial = \frac{W a^3 b^3}{3EIl^3} = \frac{500 \times 9.81 \, (0.9)^3 \, (0.6^3)}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 1.5^3}$$

$= 1.24 \times 10^{-3}$m . . . (Substituting $W = m.g$; $a = l_1$, and $b = l_2$)

We know that natural frequency of transverse vibration, $f_n = \frac{0.4985}{\sqrt{\partial}}$

$$= \frac{0.4985}{\sqrt{1.24 \times 10^{-3}}} = 14.24\text{HZ}$$

2.9. Natural Frequency of Free Transverse Vibrations Due to Uniformly Distributed Load Acting Over a Simply Supported Shaft

Consider a shaft *AB* carrying a uniformly distributed load of *w* per unit length as shown in Fig.15.

Let y_1 = Static deflection at the middle of the shaft,

a_1 = Amplitude of vibration at the middle of the shaft

$w1$ = uniformly distributed load per unit static deflection at the middle of the shaft = $w/y1$.

middle of the shaft = w/y_1.

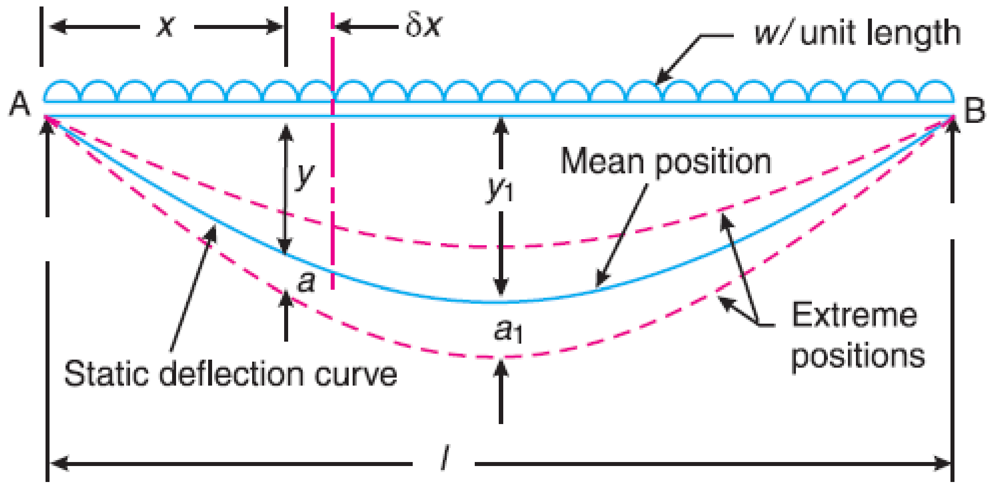


Fig.15. Simply supported shaft carrying a uniformly distributed load.

Now, consider a small section of the shaft at a distance x from A and length ∂x.

Let y = Static deflection at a distance x from A, and

a = Amplitude of its vibration.

∴ Work done on this small section

$$= \frac{1}{2} \times w_1 \times a_1 . \partial_x \times a$$

$$= \frac{1}{2} \times \frac{W}{y_1} \times a_1 \times \partial_x =$$

$$\frac{1}{2} \times w \times \frac{a_1}{y_1} \times a \times \partial_x$$

Since the maximum potential energy at the extreme position is equal to the amount of work done to move the beam from the mean position to one of its extreme positions, therefore

Maximum potential energy at the extreme position

$$= \int_0^l \frac{1}{2} \times w \times \frac{a_1}{y_1} \times a.dx \qquad \ldots (i)$$

Assuming that the shape of the curve of a vibrating shaft is similar to the static deflection curve of a beam, therefore

$$\frac{a_1}{y_1} = \frac{a}{y} = \text{Constant, } C \quad \text{or} \quad \frac{a_1}{y_1} = C \text{ and } a = y.C$$

Substituting these values in equation (*i*), we have maximum potential energy at the extreme position

$$= \int_0^l \frac{1}{2} \times w \times C \times y.C.dx = \frac{1}{2} \times w.C^2 \int_0^l y.dx \qquad \ldots (ii)$$

Since the maximum velocity at the mean position is $\omega.a_1$, where ω is the circular frequency of vibration, therefore

Maximum kinetic energy at the mean position

$$= \int_0^l \frac{1}{2} \times \frac{w.dx}{g} (\omega.a)^2 = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2.dx \qquad \ldots (iii)$$

. . .(Substituting $a = y.C$)

We know that the maximum potential energy at the extreme position is equal to the maximum kinetic energy at the mean position, therefore equating equations (*ii*) and (*iii*),

$$\frac{1}{2} \times w \times C^2 \int_0^l y.dx = \frac{w}{2g} \times \omega^2 \times C^2 \int_0^l y^2.dx$$

$$\therefore \qquad \omega^2 = \frac{g\int_0^l y.dx}{\int_0^l y^2.dx} \quad \text{or} \quad \omega = \sqrt{\frac{g\int_0^l y.dx}{\int_0^l y^2.dx}} \qquad \ldots (iv)$$

When the shaft is a simply supported, then the static deflection at a distance *x* from *A* is $y = \frac{W}{24EI}(x^4 - 2l\,x^3 + l^3 x)$ (v)

Where w = uniformly distributed load unit length,

E = Young’s modulus for the material of the shaft, and

I = Moment of inertia of the shaft.

* It has been proved in books on ‘Strength of Materials’ that maximum bending moment at a distance *x* from *A* is

$$(B.M.)_{max} = EI\frac{d^2y}{dx^2} = \frac{wx^2}{2} - \frac{wlx}{2}$$

Integrating this expression,

$$EI.\frac{dy}{dx} = \frac{wx^3}{2\times3} - \frac{wl.x^2}{2\times2} + C_1$$

On further integrating,

$$E.I.y = \frac{wx^4}{2\times3\times4} - \frac{wl.x^3}{2\times2\times3} + C_1x + C_2$$

$$= \frac{wx^4}{24} - \frac{wlx^3}{12} + C_1x + C_2$$

where C_1 and C_2 are the constants of integration and may be determined from the given conditions of the problem. Here

when $x = 0,\ y = 0$; $\therefore$ $C_2 = 0$

and when $x = l,\ y = 0$; $\therefore$ $C_1 = \frac{wl^3}{24}$

Substituting the value of C_1, we get

$$y = \frac{w}{24\,EI}(x^4 - 2lx^3 + l^3x)$$

Now integrating the above equation *(v)* within the limits from 0 to l,

$$\int_0^l y\,dx = \frac{w}{24\,EI}\int_0^l (x^4 - 2\,lx^3 + l^3x)\,dx = \frac{w}{24\,EI}\left[\frac{x^5}{5} - \frac{2\,lx^4}{4} + \frac{l^3x^2}{2}\right]_0^l$$

$$= \frac{w}{24\,EI}\left[\frac{l^5}{5} - \frac{2\,l^5}{4} + \frac{l^5}{2}\right] = \frac{w}{24\,EI}\times\frac{l^5}{5} = \frac{w.l^5}{120\,E.I} \qquad \ldots (vi)$$

Now $$\int_0^l y^2\,dx = \int_0^l \left[\frac{w}{24\,EI}(x^4 - 2lx^3 + l^3x)\right]^2 dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^8 + 4\,l^2x^6 + l^6x^2 - 4\,lx^7 - 4\,l^4x^4 + 2\,l^3x^5)\,dx$$

$$= \frac{w^2}{576\,E^2\,I^2}\cdot\left[\frac{x^9}{9} + \frac{4\,l^2x^7}{7} + \frac{l^6x^3}{3} - \frac{4\,lx^8}{8} - \frac{4\,l^4x^5}{5} + \frac{2\,l^3x^6}{6}\right]_0^l$$

$$= \frac{w^2}{576\,E^2\,I^2}\left[\frac{l^9}{9}+\frac{4\,l^9}{7}+\frac{l^9}{3}-\frac{4\,l^9}{8}-\frac{4\,l^9}{5}+\frac{2\,l^9}{6}\right]$$

$$= \frac{w^2}{576\,E^2\,I^2}\times\frac{31\,l^9}{630} \qquad \ldots (vii)$$

Substituting the value in equation *(iv)* from equations *(vi)* and *(vii)*, we get circular frequency due to uniformly distributed load,

$$\omega = \sqrt{g\left(\frac{w l^5}{120\,EI}\times\frac{576\,E^2\,I^2\times 630}{w^2\times 31\,l^9}\right)}$$

$$= \sqrt{\frac{24\,EI}{w l^4}\times\frac{630}{155}\,g} = \pi^2\sqrt{\frac{EI\,g}{w l^4}} \qquad \ldots (viii)$$

$\therefore$ Natural frequency due to uniformly distributed load,

$$f_n = \frac{\omega}{2\pi} = \frac{\pi^2}{2\pi}\sqrt{\frac{EI\,g}{w l^4}} = \frac{\pi}{2}\sqrt{\frac{EIg}{w l^4}} \qquad \ldots (ix)$$

We know that the static deflection of a simply supported shaft due to uniformly distributed load of w per unit length, is

$$\delta_S = \frac{5\,w l^4}{384\,EI} \qquad \text{or} \qquad \frac{EI}{w l^4} = \frac{5}{384\,\delta_S}$$

Equation *(ix)* may be written as

$$f_n = \frac{\pi}{2}\sqrt{\frac{5\,g}{384\,\delta_S}} = \frac{0.5615}{\sqrt{\delta_S}}\ \text{Hz} \qquad \ldots (\text{Substituting, } g = 9.81\ \text{m/s}^2)$$

2.10. Natural Frequency of Free Transverse Vibrations of a Shaft Fixed at Both Ends Carrying a Uniformly Distributed Load

Consider a shaft *AB* fixed at both ends and carrying a uniformly distributed load of *w* per unit length as shown in Fig. 2.16.

We know that the static deflection at a distance *x* from *A* is given by

$$y = \frac{W}{24EI}\,(x^4 + l^2 x^2 - 2l\,x^3) \qquad \ldots..(i)$$

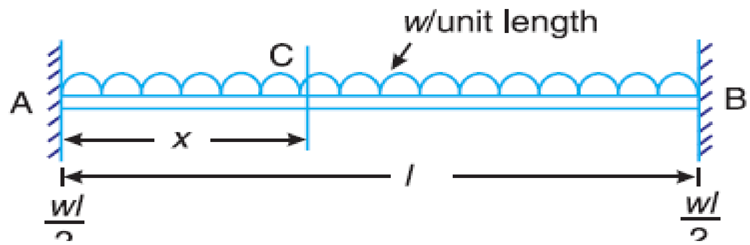


Fig.16. Shaft fixed at both carrying a uniformly distributed load.

It has been proved in books on 'Strength of Materials' that the bending moment at a distance x from A is

$$M = EI\frac{d^2y}{dx^2} = \frac{wl^2}{12} + \frac{wx^2}{2} - \frac{wlx}{2}$$

Integrating this equation,

$$EI\frac{dy}{dx} = \frac{wl^2}{12}x + \frac{wx^3}{2\times3} - \frac{wlx^2}{2\times2} + C_1$$

where C_1 is the constant of integration. We know that when $x = 0, \frac{dy}{dx} = 0$. Therefore $C_1 = 0$.

or $$EI\frac{dy}{dx} = \frac{wl^2}{12}x + \frac{wx^3}{6} - \frac{wlx^2}{4}$$

Integrating the above equation,

$$EI.y = \frac{wl^2x^2}{12\times2} + \frac{wx^4}{6\times4} - \frac{wl}{4}\times\frac{x^3}{3} + C = \frac{wl^2x^2}{24} + \frac{wx^4}{24} - \frac{wlx^3}{12} + C_2$$

where C_2 is the constant of integration. We know that when $x = 0$, $y = 0$. Therefore $C_2 = 0$.

or $$EI.y = \frac{w}{24}(l^2x^2 + x^4 - 2lx^3)$$

or $$y = \frac{w}{24\,EI}(x^4 + l^2x^2 - 2\,lx^3)$$

Integrating the above equation within limits from 0 to l,

$$\int_0^l y\,dx = \frac{w}{24\,EI}\int_0^l (x^4 + l^2x^2 - 2l\,x^3)\,dx$$

$$= \frac{w}{24\,EI}\left[\frac{x^5}{5} + \frac{l^2x^3}{3} - \frac{2l\,x^4}{4}\right]_0^l = \frac{w}{24\,EI}\left[\frac{l^5}{5} + \frac{l^5}{3} - \frac{2\,l^5}{4}\right]$$

$$= \frac{w}{24\,EI} \times \frac{l^5}{30} = \frac{wl^5}{720\,EI}$$

Now integrating y^2 within the limits from 0 to l,

$$\int_0^l y^2\,dx = \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^4 + l^2x^2 - 2l\,x^3)^2\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^8 + l^4x^4 + 4\,l^2x^6 + 2\,l^2x^6 - 4l\,x^7 - 2\,l^3x^5)\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \int_0^l (x^8 + l^4x^4 + 6\,l^2x^6 + 4\,l\,x^7 - 2\,l^3x^5)\,dx$$

$$= \left(\frac{w}{24\,EI}\right)^2 \left[\frac{x^9}{9} + \frac{l^4x^5}{5} + \frac{6\,l^2x^7}{7} - \frac{4\,l\,x^8}{8} - \frac{2\,l^3x^6}{6}\right]_0^l$$

$$= \left(\frac{w}{24\,EI}\right)^2 \left[\frac{l^9}{9} + \frac{l^9}{5} + \frac{6\,l^9}{7} - \frac{4\,l^9}{8} - \frac{2\,l^9}{6}\right] = \left(\frac{w}{24\,EI}\right)^2 \frac{l^9}{630}$$

We know that

$$\omega^2 = \frac{g\int_0^l y\,dx}{\int_0^l y^2\,dx} = g \times \frac{wl^5}{720\,EI} \times \frac{(24\,EI)^2 \times 630}{w^2 l^9} = \frac{504\,EIg}{wl^4}$$

$\therefore$

$$\omega = \sqrt{\frac{504\,EIg}{wl^4}}$$

and natural frequency,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{504\,EIg}{wl^4}} = 3.573\sqrt{\frac{E\,I\,g}{wl^4}}$$

Since the static deflection of a shaft fixed at both ends and carrying a uniformly distributed load is

$$\partial_s = \frac{Wl^4}{384\,EI} \text{ or } \frac{EI}{Wl^4} = \frac{1}{384\partial_s}$$

$$f_n = 3.573\sqrt{\frac{g}{384\partial_s}}$$

$$= \frac{0.571}{\sqrt{\partial_s}}\ HZ \qquad \ldots \text{(Substituting, } g = 9.81 \text{ m/s}^2\text{)}$$

2.11. Natural Frequency of Free Transverse Vibrations for a Shaft Subjected to a Number of Point Loads

Consider a shaft *AB* of negligible mass loaded with point loads W_1, W_2, W_3 and W_4 etc. in newton's, as shown in Fig.17. Let m_1, m_2, m_3 and m_4 etc. be the corresponding masses in kg. The natural frequency of such a shaft may be found out by the following two methods:

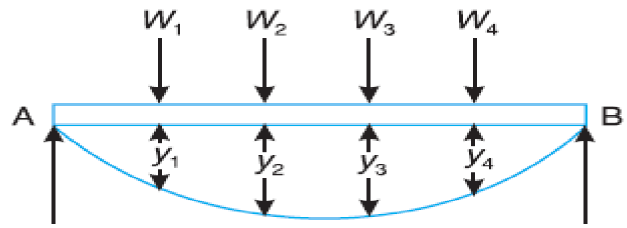


Fig.17. Shaft carrying a number of point loads.

1. *Energy (or Rayleigh's) method*

Let y_1, y_2, y_3, y_4 etc. be total deflection under loads $W1$, $W2$, $W3$ and $W4$ etc. as shown in Fig.17.

We know that maximum potential energy

$$= \frac{1}{2} \times m_1.g.y_1 + \frac{1}{2} \times m_2.g.y_2 + \frac{1}{2} m_3.g.y_3 + \frac{1}{2} \times m_4.g.y_4 + ...$$

$$= \frac{1}{2} \Sigma\, m.g.y$$

and maximum kinetic energy

$$= \frac{1}{2} \times m_1\, (\omega.y_1)^2 + \frac{1}{2} \times m_2\, (\omega.y_2)^2 + \frac{1}{2} \times m_3\, (\omega.y_3)^2 + \frac{1}{2} \times m_4\, (\omega.y_4)^2 + ...$$

$$= \frac{1}{2} \times \omega^2 \left[m_1\, (y_1)^2 + m_2\, (y_2)^2 + m_3\, (y_3)^2 + m_4\, (y_4)^2 + ... \right]$$

$$= \frac{1}{2} \times \omega^2 \Sigma\, m.y^2 \qquad \text{. . . (where } \omega = \text{Circular frequency of vibration)}$$

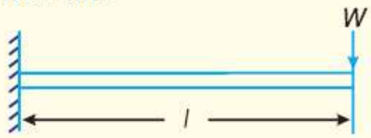
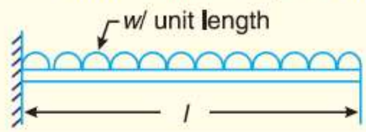
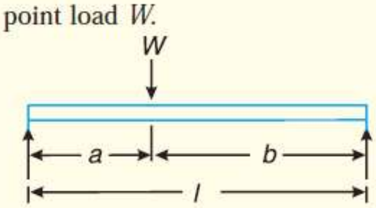
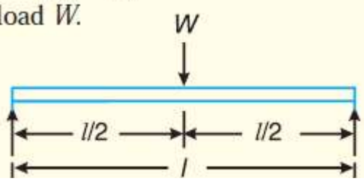
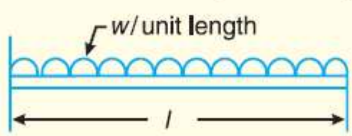
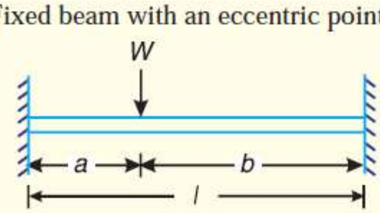
Equating the maximum kinetic energy to the maximum potential energy, we have

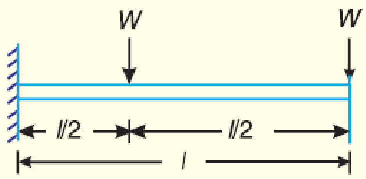
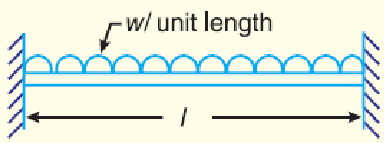
$$\frac{1}{2} \times \omega^2\, \Sigma\, m.y^2 = \frac{1}{2} \Sigma\, m.g.y$$

$$\therefore \qquad \omega^2 = \frac{\Sigma\, m.g.y}{\Sigma\, m.y^2} = \frac{g\, \Sigma\, m.y}{\Sigma\, m.y^2} \qquad \text{or} \qquad \omega = \sqrt{\frac{g\, \Sigma\, m.y}{\Sigma\, m.y^2}}$$

$\therefore$ Natural frequency of transverse vibration,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g\, \Sigma\, m.y}{\Sigma\, m.y^2}}$$

S.No.	Type of beam	Deflection (δ)
1.	Cantilever beam with a point load W at the free end. 	$\delta = \dfrac{Wl^3}{3EI}$ (at the free end)
2.	Cantilever beam with a uniformly distributed load of w per unit length. 	$\delta = \dfrac{wl^4}{8EI}$ (at the free end)
3.	Simply supported beam with an eccentric point load W. 	$\delta = \dfrac{Wa^2b^2}{3EIl}$ (at the point load)
4.	Simply supported beam with a central point load W. 	$\delta = \dfrac{Wl^3}{48EI}$ (at the centre)
5.	Simply supported beam with a uniformly distributed load of w per unit length. 	$\delta = \dfrac{5}{384} \times \dfrac{wl^4}{EI}$ (at the centre)
6.	Fixed beam with an eccentric point load W. 	$\delta = \dfrac{Wa^3b^3}{3EIl}$ (at the point load)

7.	Fixed beam with a central point load W.  	$\delta = \frac{Wl^3}{192EI}$ (at the centre)
8.	Fixed beam with a uniformly distributed load of w per unit length.  	$\delta = \frac{wl^4}{384EI}$ (at the centre)

2. *Dunkerley's method*

The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley's empirical formula. According to this

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n_1})^2} + \frac{1}{(f_{n_2})^2} + \frac{1}{(f_{n_3})^2} + \cdots + \frac{1}{((f_{n_s})^2}$$

f_n = Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load.

f_{n1}, f_{n_2}, f_{n_3}, etc. = Natural frequency of transverse vibration of each point load.

f_{n_s} = Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft).

Now, consider a shaft AB loaded as shown in Fig.18.

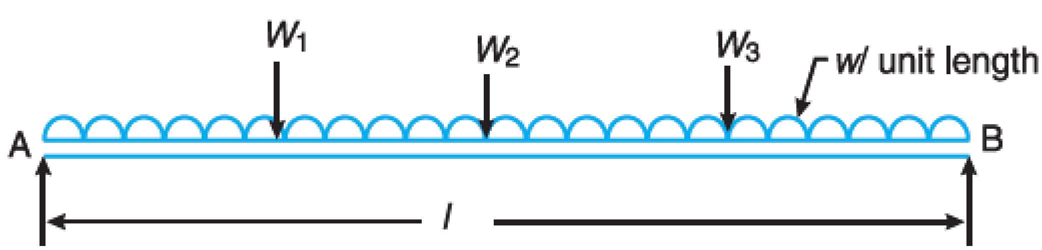


Fig.18. Shaft carrying a number of point loads and a uniformly distributed load.

Let ∂_1, ∂_2, ∂_3, etc. = Static deflection due to the load W_1, W_2, W_3 etc. when considered separately.

∂_S = Static deflection due to the uniformly distributed load or due to the mass of the shaft.
We know that natural frequency of transverse vibration due to load W_1,

$$fn_1 = \frac{0.4985}{\sqrt{\partial_1}} \text{ HZ}$$

Similarly, natural frequency of transverse vibration due to load $W2$,

$$fn_2 = \frac{0.4985}{\sqrt{\partial_2}} \text{ HZ}$$

and, natural frequency of transverse vibration due to load
$W3$,

$$fn_3 = \frac{0.4985}{\sqrt{\partial_3}} \text{ HZ}$$

Also natural frequency of transverse vibration due to uniformly distributed load or weight of the shaft, $fn_S = \frac{0.4985}{\sqrt{\partial_S}}$ HZ

Therefore, according to Dunkerley's empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n_1})^2} + \frac{1}{(f_{n_2})^2} + \frac{1}{(f_{n_3})^2} + \cdots + \frac{1}{((f_{n_S})^2}$$

$$= \frac{\partial_1}{0.4985^2} + \frac{\partial_2}{0.4985^2} + \frac{\partial_3}{0.4985^2} + \cdots + \frac{\partial_S}{0.4985^2}$$

$$= \frac{1}{(0.4985)^2}\left[\partial_1 + \partial_1 + \partial_1 \ldots + \frac{\partial_S}{1.27}\right]$$

$$f_n = \frac{0.4985}{\sqrt{\partial_1 + \partial_1 + \partial_1 \ldots + \frac{\partial_S}{1.27}}} \text{ HZ}$$

Notes: 1. when there is no uniformly distributed load or mass of the shaft is negligible, then $\partial_S = 0$

$$\therefore \quad f_n = \frac{0.4985}{\sqrt{\partial_1 + \partial_2 + \partial_3 + \cdots}}$$

2. The value of ∂_1, ∂_2, ∂_3 etc. for a simply supported shaft may be obtained from the relation $\partial = \frac{Wa^2b^2}{3EIl}$

Where ∂ = Static deflection due to load W,
a and b = Distances of the load from the ends,
E = Young's modulus for the material of the shaft,
I = Moment of inertia of the shaft, and
l = Total length of the shaft.

Example 2.4. *A shaft 50 mm diameter and 3 metres long is simply supported at the ends and carries three loads of 1000 N, 1500 N and 750 N at 1 m, 2 m and 2.5 m from the left support.*
The Young's modulus for shaft material is 200 GN/m2. Find the frequency of transverse vibration

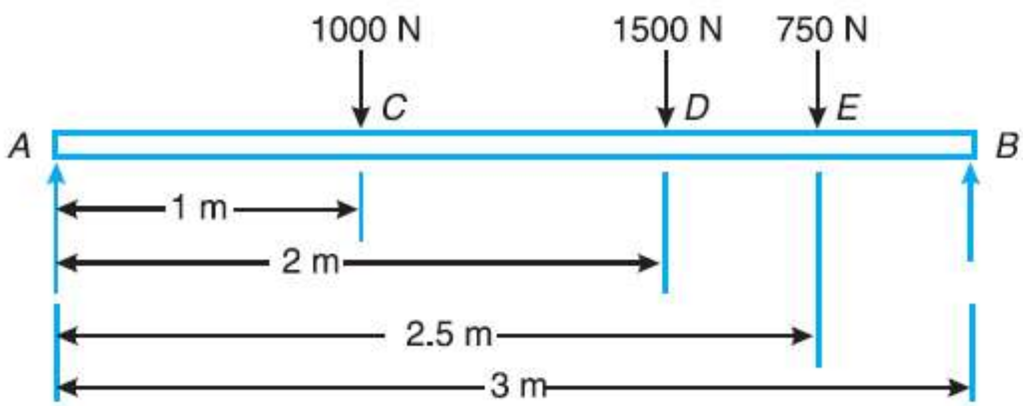


Fig. 19

Solution. Given: d = 50 mm = 0.05 m; l = 3 m, $W1$ = 1000 N; $W2$ = 1500 N ; $W3$ = 750 N; E = 200 GN/m2 = 200 × 109 N/m2

The shaft carrying the loads is shown in Fig.19

We know that moment of inertia of the shaft, $I = \frac{\pi d^4}{64} = \pi(0.05\,)^4$

$$= 0.307\times 10^{-6} m^4$$

and the static deflection due to a point load W,

$$\partial = \frac{Wa^2b^2}{3EIl}$$

∴ Static deflection due to a load of 1000 N,

$$\partial_1 = \frac{1000\times 1^2\times 2^2}{3\times 200\times 10^9\times 0.307\times 10^{-6}\times 3}$$

$$= 7.24\times 10^{-3} m \quad \ldots \text{(Here } a = 1 \text{ m, and } b = 2 \text{ m)}$$

Similarly, static deflection due to a load of 1500 N,

$$\delta_2 = \frac{1500 \times 2^2 \times 1^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 10.86 \times 10^{-3} \text{ m}$$

. . . (Here a = 2 m, and b = 1 m)

and static deflection due to a load of 750 N,

$$\delta_3 = \frac{750\,(2.5)^2\,(0.5)^2}{3 \times 200 \times 10^9 \times 0.307 \times 10^{-6} \times 3} = 2.12 \times 10^{-3} \text{ m}$$

. . . (Here a = 2.5 m, and b = 0.5 m)

We know that frequency of transverse vibration,

$$f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3}} = \frac{0.4985}{\sqrt{7.24 \times 10^{-3} + 10.86 \times 10^{-3} + 2.12 \times 10^{-3}}}$$

$$= \frac{0.4985}{0.1422} = 3.5 \text{ Hz } \textbf{Ans.}$$

2.12. Frequency of Free Damped Vibrations (Viscous Damping)

We have already discussed that the motion of a body is resisted by frictional forces. In vibrating systems, the effect of friction is referred to as damping. The damping provided by fluid resistance is known as ***viscous damping*****.**

We have also discussed that in damped vibrations, the amplitude of the resulting vibration gradually diminishes. This is due to the reason that a certain amount of energy is always dissipated to overcome the frictional resistance. The resistance to the motion of the body is provided partly by the medium in which the vibration takes place and partly by the internal friction, and in some cases partly by a dash pot or other external damping device.

Consider a vibrating system, as shown in Fig.20, in which a mass is suspended from one end of the spiral spring and the other end of which is fixed. A damper is provided between the mass and the rigid support.

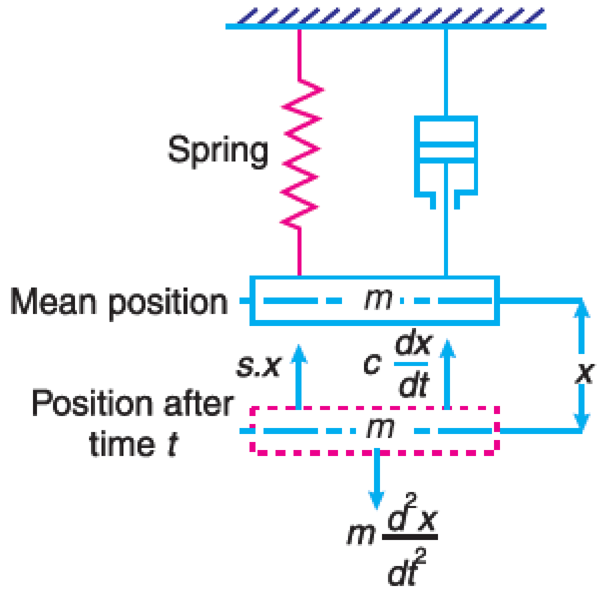


Fig. 20. Frequency of free damped vibrations.

Let $\quad$ ***m*** = Mass suspended from the spring,

s = Stiffness of the spring,

x = Displacement of the mass from the mean position at time *t*,

∂ = Static deflection of the spring

= *m.g/s*, and

c = Damping coefficient or the damping force per unit velocity.

Since in viscous damping, it is assumed that the frictional resistance to the motion of the body is directly proportional to the speed of the movement, therefore Damping force or frictional force on the mass acting in ***opposite*** direction to the motion of the mass

$$= C \times \frac{dx}{dt}$$

Accelerating force on the mass, acting ***along*** the motion of the mass

$$= \frac{md^2x}{dt^2}$$

Accelerating force on the mass, acting ***along*** the motion of the mass

$$= s.x$$

Therefore the equation of motion becomes

$$\frac{m\ d^2x}{dt^2} = -\left(c\times \frac{dx}{dt} + s.x\right)$$

(Negative sign indicates that the force opposes the motion)

$$\text{Or} \quad \frac{m\ d^2x}{dt^2} + C \times \frac{dx}{dt} + s.x = 0$$

Or $\frac{d^2x}{dt^2} + \frac{c}{m} + \frac{dx}{dt} + \frac{s}{m} \times x = 0$

This is a differential equation of the second order. Assuming a solution of the form $x = e^{kt}$ where k is a constant to be determined. Now the above differential equation reduces to

$$k^2.e^{kt} + \frac{c}{m} \times k.e^{kt} + \frac{s}{m} \times e^{kt} = 0 \qquad \ldots \left[\because \frac{dx}{dt} = ke^{kt}, \text{ and } \frac{d^2x}{dt^2} = k^2.e^{kt} \right]$$

or $$k^2 + \frac{c}{m} \times k + \frac{s}{m} = 0 \qquad \ldots (i)$$

and $$k = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4 \times \frac{s}{m}}}{2}$$

$$= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

$\therefore$ The two roots of the equation are

$$k_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

and $$k_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}$$

The most general solution of the differential equation ***(i)*** with its right hand side equal to zero has only complementary function and it is given by

$$X_1 = C_1 e^{k_1 t} + C_1 e^{k_2 t} \qquad \ldots (ii)$$

Where C1 and *C*2 are two arbitrary constants which are to be determined from the initial conditions of the motion of the mass.

It may be noted that the roots k_1 and k_2 may be real, complex conjugate (imaginary) or equal. We shall now discuss these three cases as below:

* A system described by this equation is said to be a single degree of freedom harmonic oscillator with viscous damping.

1. *When the roots are real (overdamping)*

If $\left(\frac{c}{2m}\right)^2 > \frac{s}{m}$, then the roots k_1 and k_2 are real but negative. This is a case of ***overdamping*** or ***large damping*** and the mass moves slowly to the equilibrium position. This motion is known as ***aperiodic.*** When the roots are real, the most general solution of the differential equation is

$$x = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

$$= C_1\, e^{\left[-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t} + C_2\, e^{\left[-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{s}{m}}\right]t}$$

Note : In actual practice, the overdamped vibrations are avoided.

2. *When the roots are complex conjugate (underdamping)*

If $\frac{s}{m} > \left(\frac{c}{2m}\right)^2$, then the radical (*i.e.* the term under the square root) becomes negative. The two roots k_1 and k_2 are then known as complex conjugate. This is a most practical case of damping and it is known as ***underdamping*** or ***small damping.*** The two roots are

$$k_1 = -\frac{c}{2m} + i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

and $$k_2 = -\frac{c}{2m} - i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

where i is a Greek letter known as iota and its value is $\sqrt{-1}$. For the sake of mathematical calculations, let

$$\frac{c}{2m} = a; \ \frac{s}{m} = (\omega_n)^2; \text{ and } \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \omega_d = \sqrt{(\omega_n)^2 - a^2}$$

Therefore the two roots may be written as

$$k_1 = -a + i\omega_d \ ; \quad \text{and} \quad k_2 = -a - i\omega_d$$

We know that the general solution of a differential equation is

$$x = C_1\, e^{k_1 t} + C_2\, e^{k_2 t} = C_1\, e^{(-a + i\omega_d)t} + C_2\, e^{(-a - i\omega_d)t}$$

$$= e^{-at}\left(C_1\, e^{i\omega_d \cdot t} + C_2\, e^{-i\omega_d t}\right) \quad \ldots(\text{Using } e^{m+n} = e^m \times e^n) \ \ldots(iii)$$

Now according to Euler's theorem

$$e^{+i\theta} = \cos\theta + i\sin\theta \ ; \text{ and } e^{-i\theta} = \cos\theta - i\sin\theta$$

Therefore the equation ***(iii)*** may be written as

$$x = e^{-at}\left[C_1(\cos\omega_d . t + i\sin\omega_d . t) + C_2(\cos\omega_d . t - i\sin\omega_d . t)\right]$$

$$= e^{-at}\left[(C_1 + C_2)\cos\omega_d . t + i(C_1 - C_2)\sin\omega_d . t\right]$$

and $$k_2 = -\frac{c}{2m} - i\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$

where i is a Greek letter known as iota and its value is $\sqrt{-1}$. For the sake of mathematical calculations, let

$$\frac{c}{2m} = a; \ \frac{s}{m} = (\omega_n)^2; \text{ and } \sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} = \omega_d = \sqrt{(\omega_n)^2 - a^2}$$

Therefore the two roots may be written as

$$k_1 = -a + i\omega_d \ ; \quad \text{and} \quad k_2 = -a - i\omega_d$$

We know that the general solution of a differential equation is

$$x = C_1\, e^{k_1 t} + C_2\, e^{k_2 t} = C_1\, e^{(-a + i\omega_d)t} + C_2\, e^{(-a - i\omega_d)t}$$

$$= e^{-at}(C_1 e^{i\omega_d . t} + C_2 e^{-i\omega_d t}) \quad \ldots(\text{Using } e^{m+n} = e^m \times e^n) \ldots (iii)$$

Now according to Euler's theorem

$$e^{+i\theta} = \cos\theta + i\sin\theta \; ; \text{ and } \; e^{-i\theta} = \cos\theta - i\sin\theta$$

Therefore the equation ***(iii)*** may be written as

$$x = e^{-at}\left[C_1(\cos\omega_d .t + i\sin\omega_d .t) + C_2(\cos\omega_d .t - i\sin\omega_d .t)\right]$$

$$= e^{-at}\left[(C_1 + C_2)\cos\omega_d .t + i(C_1 - C_2)\sin\omega_d .t)\right]$$

Let $C_1 + C_2 = A,$ and $i(C_1 - C_2) = B$

$$\therefore \quad x = e^{-at}(A\cos\omega_d .t + B\sin\omega_d .t) \qquad \ldots (iv)$$

Again, let $A = C\cos\theta$, and $B = C\sin\theta$, therefore

$$C = \sqrt{A^2 + B^2} \;, \text{ and } \quad \tan\theta = \frac{B}{A}$$

Now the equation ***(iv)*** becomes

$$x = e^{-at}(C\cos\theta\cos\omega_d .t + C\sin\theta\sin\omega_d .t)$$

$$= Ce^{-at}\cos(\omega_d .t - \theta) \qquad \ldots (v)$$

If t is measured from the instant at which the mass m is released after an initial displacement A, then

$$A = C\cos\theta \qquad \ldots [\text{Substituting } x = A \text{ and } t = 0 \text{ in equation } (v)]$$

and when $\theta = 0$, then $A = C$

$\therefore$ The equation ***(v)*** may be written as

$$x = Ae^{-at}\cos\omega_d .t \qquad \ldots (vi)$$

where $\omega_d = \sqrt{\dfrac{s}{m} - \left(\dfrac{c}{2m}\right)^2} = \sqrt{(\omega_n)^2 - a^2}$; and $a = \dfrac{c}{2m}$

We see from equation ***(vi)***, that the motion of the mass is simple harmonic whose circular damped frequency is ω_d and the amplitude is Ae^{-at} which diminishes exponentially with time as shown in Fig.21. Though the mass eventually returns to its equilibrium position because of its inertia, yet it overshoots and the oscillations may take some considerable time to die away,

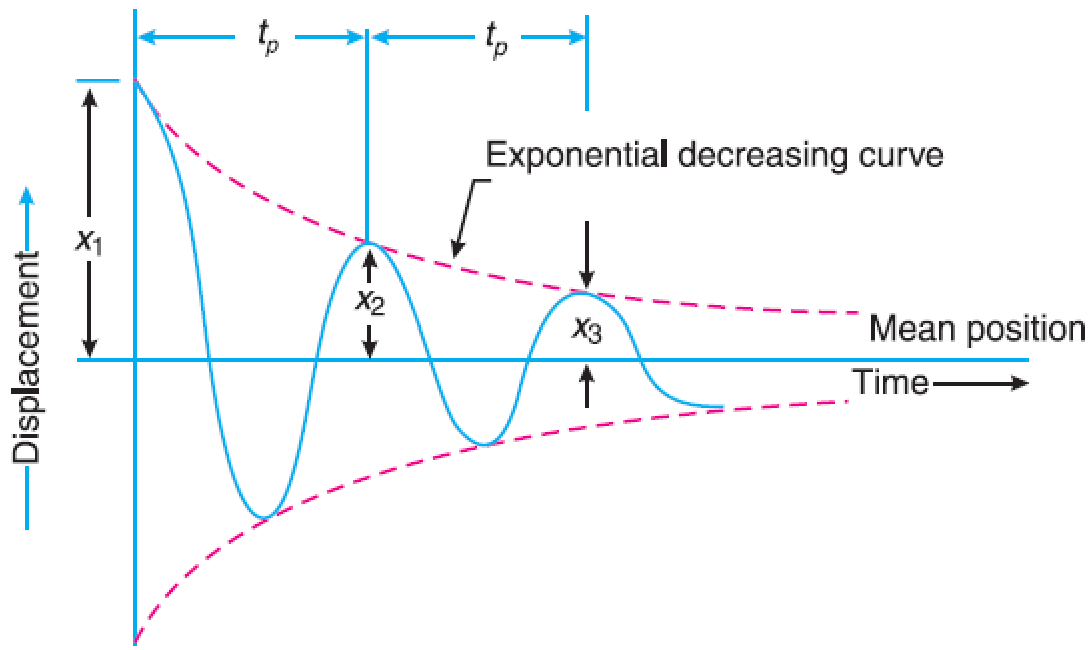


Fig. 21 Underdamping or small damping.

We know that the periodic time of vibration,

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

and frequency of damped vibration,

$$f_d = \frac{1}{t_p} = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2} = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2} \quad \text{. . . } (vii)$$

Note: When no damper is provided in the system, then $c = 0$. Therefore the frequency of the undamped vibration,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

. . . [Substituting $c = 0$, in equation ***(vii)***]

It is the same as discussed under free vibrations.

3. *When the roots are equal (critical damping)*

If $\left(\frac{c}{m}\right)^2 = \frac{s}{m}$, then the radical becomes zero and the two roots k_1 and k_2 are equal. This is a case of **critical damping.** In other words, the critical

damping is said to occur when frequency of damped vibration (f_d) is zero (*i.e.* motion is aperiodic). This type of damping is also avoided because the mass moves back rapidly to its equilibrium position, in the shortest possible time.
For critical damping, equation ***(ii)*** may be written as

$$x = (C_1 + C_2)\, e^{-\frac{c}{2m}t} = (C_1 + C_2)\, e^{-\omega_n t} \qquad \ldots \left[\because \frac{c}{2m} = \sqrt{\frac{s}{m}} = \omega_n\right]$$

Thus the motion is again aperiodic. The critical damping coefficient (c_c) may be obtained by substituting c_c for c in the condition for critical damping, *i.e.*

$$\left(\frac{c_c}{2m}\right)^2 = \frac{s}{m} \quad \text{or} \quad c_c = 2m\sqrt{\frac{s}{m}} = 2m \times \omega_n$$

The critical damping coefficient is the amount of damping required for a system to be critically damped.

2.13. Damping Factor or Damping Ratio

The ratio of the actual damping coefficient (c) to the critical damping coefficient (C_c) is known as ***damping factor* or *damping ratio*.** Mathematically,

Damping factor $= \dfrac{c}{c_c} = \dfrac{c}{2.m\,\omega_n}$ $\quad \ldots (\because\ Cc = 2\pi\omega_n)$

The damping factor is the measure of the relative amount of damping in the existing system with that necessary for the critical damped system.

2.14. Logarithmic Decrement

It is defined as the natural logarithm of the amplitude reduction factor. The amplitude reduction factor is the ratio of any two successive amplitudes on the same side of the mean position.
If x_1 and x_2 are successive values of the amplitude on the same side of the mean position, as shown in Fig.22, then amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{Ae^{-at}}{Ae^{-a(t+t_p)}} = e^{at_p} \quad = \text{constant}$$

where t_p is the period of forced oscillation or the time difference between two consecutive amplitudes. As per definition, logarithmic decrement,

$$\delta = \log\left(\frac{x_1}{x_2}\right) = \log e^{at_p}$$

or

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = a.t_p = a \times \frac{2\pi}{\omega_d} = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\ldots \left[\because \ \omega_d = \sqrt{(\omega_n)^2 - a^2}\right]$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\sqrt{(\omega_n)^2 - \left(\frac{c}{2m}\right)^2}} \qquad \ldots \left(\because \ a = \frac{c}{2m}\right)$$

$$= \frac{\frac{c}{2m} \times 2\pi}{\omega_n \sqrt{1 - \left(\frac{c}{2m.\omega_n}\right)^2}} = \frac{c \times 2\pi}{c_c \sqrt{1 - \left(\frac{c}{c_c}\right)^2}} \qquad \ldots (\because \ c_c = 2m.\omega_n)$$

In general, amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots. = \frac{x_n}{x_{n+1}} = e^{at_p} = \text{constant}$$

$\therefore$ Logarithmic decrement,

$$\partial = log_e\left(\frac{x_n}{x_{n+1}}\right) = a.t_p = \frac{2\pi c}{\sqrt{(C_C)^2 - C^2}}$$

Example 2.5 *A vibrating system consists of a mass of 200 kg, a spring of stiffness 80 N/mm and a damper with damping coefficient of 800 N/m/s. Determine the frequency of vibration of the system.*

Solution. Given: m = 200 kg; s = 80 N/mm = 80 × 103 N/m; c = 800 N/m/s

We know that circular frequency of undamped vibrations,

$$w_n = \frac{s}{m} = \sqrt{\frac{80 \times 10^3}{200}} = 20 \text{ rad/s}$$

And circular frequency of damped vibrations,

$$w_d = \sqrt{(w_n)^2 - a^2} = \sqrt{(w_n)^2 - (\frac{c}{2m})^2} \quad \ldots\ldots\ldots (\because\ a = c/2m)$$

$$= \sqrt{(20)^2 - \left(\frac{800}{2} \times 200\right)^2} = 19.9 \text{rad/s}$$

∴ Frequency of vibration of the system,

$$f_d = \omega_d / 2\pi = 19.9/2\pi \quad 3.17 \text{ Hz } \textbf{Ans.}$$

Example 2.6. *The following data are given for a vibratory system with viscous damping:*
Mass = 2.5 kg; spring constant = 3 N/mm and the amplitude decreases to 0.25 of the initial value after five consecutive cycles.
Determine the damping coefficient of the damper in the system.

Solution. Given: m = 2.5 kg; s = 3 N/mm = 3000 N/m; $x_6 = 0.25\, x_1$

We know that natural circular frequency of vibration,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{3000}{2.5}} = 34.64 \text{ rad/s}$$

Let c = Damping coefficient of the damper in N/m/s,

x_1 = Initial amplitude, and

x_6 = Final amplitude after five consecutive cycles = $0.25\, x_1$ …(Given)

We know that

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}$$

or

$$\frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left(\frac{x_1}{x_2}\right)^5$$

∴

$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\, x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$log_e(1.32) = a \times \frac{2\pi}{\sqrt{((34.64)^2 - a^2)}} \quad \text{or } 0.2776 = a \times 2\pi/(1200 - a^2)$$

Squaring both sides, $\quad 0.077 = \dfrac{39.5a^2}{1200 - a^2}$ or $9.24 - 0.077a^2$

$= 39.5a^2$; $a^2 = 2.335$ or $a = 1.53$

We know that $a = C/2m$ or $C = a \times 2m$

$= 1.53 \times 2 \times 2.5 = 7.65\ N/\text{m/s}$

Example 2.7. *An instrument vibrates with a frequency of 1 Hz when there is no damping.*
When the damping is provided, the frequency of damped vibrations was observed to be 0.9 Hz.
Find **1.** *The damping factor, and* **2.** *Logarithmic decrement.*

Solution. Given: $f_n = 1$ Hz; $f_d = 0.9$ Hz

1. *Damping factor*

Let m = Mass of the instrument in kg,
c = Damping coefficient or damping force per unit velocity in N/m/s, and
c_c= Critical damping coefficient in
N/m/s.

We know that natural circular frequency of undamped vibrations,

$\omega_n = 2\pi \cdot f_n = 2\pi \times 1 = 6.284$ rad/s

And circular frequency of damped vibrations,

$\omega_d = 2\pi \times fd = 2\pi \times 0.9 = 5.66$ rad/s

We also know that circular frequency of damped vibrations (ωd),

We also know that circular frequency of damped vibrations (ω_d),

$$5.66 = \sqrt{(\omega_n)^2 - a^2} = \sqrt{(6.284)^2 - a^2}$$

Squaring both sides,

$$(5.66)^2 = (6.284)^2 - a^2 \text{ or } 32 = 39.5 - a^2$$

$\therefore$ $a^2 = 7.5$ or $a = 2.74$

We know that, $a = c/2m$ or $c = a \times 2m = 2.74 \times 2m = 5.48\ m$ N/m/s

$\therefore$ Damping factor,

$$c/c_c = 5.48m/12.568m = 0.436 \textbf{ Ans.}$$

2. *Logarithmic decrement*

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi \times 5.48\,m}{\sqrt{(12.568\,m)^2 - (5.48\,m)^2}} = \frac{34.4}{11.3} = 3.04 \textbf{ Ans.}$$

Example 2.8. *The measurements on a mechanical vibrating system show that it has a mass of 8 kg and that the springs can be combined to give an equivalent spring of stiffness*
5.4 N/mm. If the vibrating system have a dashpot attached which exerts a force of 40 N when the mass has a velocity of 1 m/s, find: **1.** *critical damping coefficient,* **2.** *damping factor,* **3.** *Logarithmic decrement, and* **4.** *ratio of two consecutive amplitudes.*

Solution. Given: m = 8 kg; s = 5.4 N/mm = 5400 N/m

Since the force exerted by dashpot is 40 N, and the mass has a velocity of 1 m/s, therefore

Damping coefficient (actual), c = 40 N/m/s

1. *Critical damping coefficient*

We know that critical damping coefficient,

$$c_c = 2m\omega_n = 2m\times\sqrt{\frac{s}{m}} = 2\times 8\sqrt{\frac{5400}{8}} = 416 \text{ N/m/s Ans}$$

2. *Damping factor*

We know that damping factor

$$= \frac{c}{c_c} = \frac{40}{416} = 0.096 \text{ Ans.}$$

3. *Logarithmic decrement*

We know that logarithmic decrement,

$$\delta = \frac{2\pi c}{\sqrt{(c_c)^2 - c^2}} = \frac{2\pi\times 40}{\sqrt{(416)^2 - (40)^2}} = 0.6 \text{ Ans.}$$

4. *Ratio of two consecutive amplitudes*

Let x_n and x_{n+1} = Magnitude of two consecutive amplitudes,

We know that logarithmic decrement,

$$\delta = \log_e\left[\frac{x_n}{x_{n+1}}\right] \text{ or } \frac{x_n}{x_{n+1}} = e^{\delta} = (2.7)^{0.6} = 1.82 \text{ Ans.}$$

We know that $log_e\,(\frac{x_1}{x_2})$ =a.t$_p$ or $log_e(\frac{x_1}{0.2_{x_1}})$

= a× 0.67

$log_e\,0.5$ = 0.67a or 1.61 = 0.67a or a = 2.4 …..($\therefore$ $\log 5_e = 1.61$)

We also know that frequency of free damped vibration,

$$f_d = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2}$$

or $$(\omega_n)^2 = (2\pi \times f_d)^2 + a^2 \qquad \text{. . . (By squaring and arranging)}$$

$$= (2\pi \times 1.5)^2 + (2.4)^2 = 94.6$$

∴ $$\omega_n = 9.726 \text{ rad/s}$$

We know that frequency of undamped vibration,

$$f_n = \frac{\omega_n}{2\pi} = \frac{9.726}{2\pi} = 1.55 \text{ Hz } \textbf{Ans.}$$

Example 2.9. *A coil of spring stiffness 4 N/mm supports vertically a mass of 20 kg at the free end. The motion is resisted by the oil dashpot. It is found that the amplitude at the beginning of the fourth cycle is 0.8 times the amplitude of the previous vibration. Determine the damping force per unit velocity. Also find the ratio of the frequency of damped and undamped vibrations.*

Solution. Given: s = 4 N/mm = 4000 N/m; m = 20 kg

Damping force per unit velocity

Let c = Damping force in newtons per unit velocity *i.e.* in N/m/s

x_n = Amplitude at the beginning of the third cycle,

x_{n+1} = Amplitude at the beginning of the fourth cycle = 0.8 x_n . . . (Given)

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{4000}{20}} = 14.14 \text{ rad/s}$$

and $$\log_e\left(\frac{x_n}{x_n + 1}\right) = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

or $$\log_e\left(\frac{x_n}{0.8\,x_n}\right) = a \times \frac{2\pi}{\sqrt{(14.14)^2 - a^2}}$$

$$\log_e 1.25 = a \times \frac{2\pi}{\sqrt{200 - a^2}} \quad \text{or} \quad 0.223 = a \times \frac{2\pi}{\sqrt{200 - a^2}}$$

Squaring both sides

$$0.05 = \frac{a^2 \times 4\,\pi^2}{200 - a^2} = \frac{39.5 \times a^2}{200 - a^2}$$

a^2 = 10/39.5a^2= 0.25 or a = 0.5

We know that a = c/2m, then c = a×2m = 0.5×2×20 = 20 N/m/s

Ratio of the frequencies

Let f_{n1} = Frequency of damped vibrations = $\frac{\omega_d}{2\pi}$

f_{n2} =Frequency of undamped vibrations = $\frac{\omega_n}{2\pi}$

$$\frac{fn_1}{fn_2} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega n} = \frac{\omega_d}{\omega_n} = \sqrt{\frac{(\omega_n)^{2} - a^2}{\omega n}}$$

$$= \sqrt{\frac{(14.14)^2 - (0.5)^2}{14.14}} \quad \ldots (\because w_d = \sqrt{(w_n)^{2} - a^2}) = 0.999$$

Example 2.10. *A machine of mass 75 kg is mounted on springs and is fitted with a dashpot to damp out vibrations. There are three springs each of stiffness 10 N/mm and it is found that the amplitude of vibration diminishes from 38.4 mm to 6.4 mm in two complete oscillations.*

Assuming that the damping force varies as the velocity, determine: ***1.*** *the resistance of the dashpot at unit velocity;* ***2.*** *the ratio of the frequency of the damped vibration to the frequency of the undamped vibration; and* ***3.*** *the periodic time of the damped vibration.*

Solution. Given: m = 75 kg; s = 10 N/mm = 10 $\times 10^3$ N/m; x_1 = 38.4 mm = 0.0384 m; x_3 = 6.4 mm = 0.0064 m

Since the stiffness of each spring is 10×10^3 N/m and there are 3 springs, therefore total stiffness,

$$s = 3\times10\times10^3 = 30\times10^3 \text{ N/m}$$

We know that natural circular frequency of motion,

$$\omega_n = \sqrt{\frac{s}{m}} -= \sqrt{\frac{30\times10^3}{75}} = 20\text{rad/s}$$

1. *Resistance of the dashpot at unit velocity*

Let c = Resistance of the dashpot in newtons at unit velocity *i.e.* in N/m/s,

x_2 = Amplitude after one complete oscillation in metres, and

x_3 = Amplitude after two complete oscillations in metres.

We know that

$$\therefore \left[\frac{x_1}{x_2}\right]^2 = \frac{x_1}{x_3} \quad \ldots\left[\because \frac{x_1}{x_3} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} = \frac{x_1}{x_2} \times \frac{x_1}{x_2} = \left[\frac{x_1}{x_2}\right]^2\right]$$

Or $$\frac{x_1}{x_2} = \left[\frac{x_1}{x_3}\right]^{1/2} = \left(\frac{0.0384}{0.0064}\right)^{1/2} = 2.45$$

We also know that

$$log_e = \frac{x_1}{x_2} = a \times \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$log\ 2.45 = a \times \frac{2\pi}{\sqrt{(20)^2 - a^2}} = 0.8951 = \frac{a \times 2\pi}{\sqrt{400 - a^2}}$$

$$or\ 0.8 = \frac{a^2 \times 39.5}{400 - a^2} \quad \ldots\ldots\ldots \text{(Squaring both sides)}$$

$\therefore$ $a^2 = 7.94$ or $a = 2.8$

We know that $a = c / 2m$

$\therefore c = a \times 2m = 2.8 \times 2 \times 75 = 420$ N/m/s **Ans.**

2. Ratio of the frequency of the damped vibration to the frequency of undamped vibration

Let f_{n1} = Frequency of damped vibration = $\frac{\omega_d}{2\pi}$

f_{n2} = Frequency of undamped vibration = $\frac{\omega_n}{2\pi}$

$$\therefore \quad \frac{f_{n1}}{f_{n2}} = \frac{\omega_d}{2\pi} \times \frac{2\pi}{\omega_n} = \frac{\omega_d}{\omega_n} = \frac{\sqrt{(\omega_n)^2 - a^2}}{\omega_n} = \frac{\sqrt{(20)^2 - (2.8)^2}}{20} = 0.99 \text{ Ans.}$$

3. Periodic time of damped vibration

We know that periodic time of damped vibration

$$= \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{2\pi}{\sqrt{(20)^2 - (2.8)^2}} = 0.32 \text{ s Ans.}$$

Example 2.11. *The mass of a single degree damped vibrating system is 7.5 kg and makes 24 free oscillations in 14 seconds when disturbed from its equilibrium position. The amplitude of vibration reduces to 0.25 of its initial value after five oscillations. Determine:* ***1.*** *stiffness of the spring,* ***2.*** *logarithmic decrement, and* ***3.*** *damping factor, i.e. the ratio of the system damping to critical damping.*

Solution. Given: $m = 7.5$ kg

Since 24 oscillations are made in 14 seconds, therefore frequency of free vibrations,

$fn = 24/14 = 1.7$

and $\omega_n = 2 \cdot \pi \times fn = 2\pi \times 1.7 = 10.7$ rad/s

1. *Stiffness of the spring*

Let s = Stiffness of the spring in N/m.

We know that $(\omega_n)^2 = s/m$ or $s = (\omega_n)^2 m$ = $(10.7)^2 \times 7.5$ = 860 N/m **Ans.**

2. *Logarithmic decrement*

Let $x1$ = Initial amplitude,

$X6$ = Final amplitude after five oscillations = 0.25 $x1$ ... (Given)

$$\therefore \quad \frac{x_1}{x_6} = \frac{x_1}{x_2} \times \frac{x_2}{x_3} \times \frac{x_3}{x_4} \times \frac{x_4}{x_5} \times \frac{x_5}{x_6} = \left[\frac{x_1}{x_2}\right]^5$$

$$\text{.........} \left[\therefore \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = \frac{x_5}{x_6}\right]$$

or
$$\frac{x_1}{x_2} = \left(\frac{x_1}{x_6}\right)^{1/5} = \left(\frac{x_1}{0.25\,x_1}\right)^{1/5} = (4)^{1/5} = 1.32$$

We know that logarithmic decrement,

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = \log_e 1.32 = 0.28 \text{ Ans.}$$

3. *Damping factor*

Let c = Damping coefficient for the actual system, and

c_c = Damping coefficient for the critical damped system.

We know that logarithmic decrement (δ),

$$0.28 = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} = \frac{a \times 2\pi}{\sqrt{(10.7)^2 - a^2}}$$

$$0.0784 = \frac{a^2 \times 39.5}{114.5 - a^2} \qquad \text{. . . (Squaring both sides)}$$

$$8.977 - 0.0784\,a^2 = 39.5\,a^2 \quad \text{or} \quad a^2 = 0.227 \quad \text{or} \quad a = 0.476$$

We know that $a = c/2m$ or $c = a \times 2m = 0.476 \times 2 \times 7.5 = 7.2$ N/m/s **Ans.**

and $c_c = 2m.\omega_n = 2 \times 7.5 \times 10.7 = 160.5$ N/m/s **Ans.**

∴ Damping factor = c/c_c = 7.2 / 160.5 = 0.045 **Ans.**